TATA66 Exercises, fifth set

Hand in solutions to six of the following ten exercises

1 Let $\varphi = \chi_{[0,1[}$ be the Haar scaling function and $\psi = \chi_{[0,1/2[} - \chi_{[1/2,1[}$ the Haar wavelet.

(a) Determine the coefficients in the expansion $\varphi = \sum_{j,k\in\mathbb{Z}} c_{jk}\psi_{jk}$ in $L^2(\mathbb{R})$.

(b) Find the error in the following calculation:

$$1 = \int_{\mathbb{R}} \varphi(x)\,dx = \int_{\mathbb{R}} \sum_{j,k\in\mathbb{Z}} c_{jk}\psi_{jk}(x)\,dx = \sum_{j,k\in\mathbb{Z}} c_{jk}\int_{\mathbb{R}} \psi_{jk}(x)\,dx = \sum_{j,k\in\mathbb{Z}} c_{jk}\cdot 0 = 0.$$

2 Let W_j be the detail spaces of an MRA. Show that $f(x) \in W_j \iff f(2x) \in W_{j+1}$.

***3** Show that if $\varphi \in L^1(\mathbb{R})$ satisfies $\varphi(x) = \varphi(2x) + \varphi(2x-1)$ almost everywhere, then there exists a constant C such that $\varphi(x) = C\chi_{[0,1[}(x)$ almost everywhere.

4 Let $\Phi(x) = (1+\cos\pi x)\chi_{[-1,1]}(x)$. Determine $\sum_{k\in\mathbb{Z}} \left|\widehat{\Phi}(\xi+2\pi k)\right|^2$ and use the result to show that $(\Phi_{0k})_{k\in\mathbb{Z}}$ is a Riesz system in $L^2(\mathbb{R})$.

5 Let c_k be the structure constants for φ, a scaling function for an MRA. Show that

$$\sum_{k\in\mathbb{Z}} c_k\overline{c_{2n+k}} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

6 Suppose that a scaling function φ for an MRA belongs to $L^1(\mathbb{R})$ and that only finitely many of the structure constants c_k for φ are nonzero. Show that $\widehat{\varphi}(0) \neq 0$ and that $\sum_{k\in\mathbb{Z}} c_k = \sqrt{2}$.

***7** Prove that if φ is a scaling function for an MRA and ψ is the corresponding wavelet, then, for almost every $\xi \in \mathbb{R}$,

$$|\widehat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}(2^j\xi)|^2.$$

8 Use any tools you like (computer programs, presumably) to create a reasonably accurate picture of the scaling function φ that has compact support, satisfies $\widehat{\varphi}(0) = 1$, and has filter $m(\xi) = (2 + 6e^{-i\xi} + 3e^{-i2\xi} - e^{-i3\xi})/10$. You don't need to document your work, just hand in the picture.

***9** Determine one version of the filter m_1 from the last lecture explicitly, as follows:

(a) Let $g_1(\xi) = 1 - (1/a_1)\int_0^\xi \sin^3\eta\,d\eta$, where $a_1 = \int_0^\pi \sin^3\eta\,d\eta$. Rewrite g_1 in the form $g_1(\xi) = \sum_{k=-3}^{3}\gamma_k e^{ik\xi}$.

(b) Construct $m_1(\xi) = (1/\sqrt{2})\sum_{k=0}^{3} c_k e^{-ik\xi}$ such that $|m_1(\xi)|^2 = g_1(\xi)$ and such that $m_1(0) = 1$, by using the method in the proof of Riesz's lemma (pick roots on or inside the unit circle when constructing m_1), and then making a final adjustment.

***10** Give an example of a scaling function φ for an MRA such that φ does not have compact support and such that only finitely many of the structure constants for φ are nonzero.