AN INTRODUCTION TO WAVELETS

BENGT OVE TURESSON

Contents

1. Introduction

Consider the Fourier expansion of a function $f \in L^2(0,1)$:

$$f(x) = \sum_{k \in \mathbf{Z}} c_k e^{2\pi i k x},$$

where the series converges in $L^2(0,1)$ and the coefficients c_k are given by

$$c_k = \int_0^1 f(x) e^{-2\pi i k x}\, dx, \quad k \in \mathbf{Z}.$$

The basis functions $e^{2\pi i k x}$ are perfectly localized with respect to frequency and the Fourier expansion is therefore a very useful tool for studying problems that concern frequency. If we interpret f as a signal, we may for instance find out what the dominating frequencies in the signal are by comparing the size of the coefficients. Assuming that the coefficients c_k in the expansion decay rapidly, the signal may furthermore be compressed effectively by ignoring coefficients corresponding to high frequencies. We can also find out what frequencies contribute most to the energy content of the signal from Parseval's identity:

$$\sum_{k \in \mathbf{Z}} |c_k|^2 = \int_0^1 |f(x)|^2\, dx.$$

Here, the right-hand side is the energy of the signal f.

The functions in the basis are, however, not at all localized with respect to x, that is, with respect to space or time, depending on what meaning we give to the

Date: January 28, 2015.

variable x. A consequence of this fact is that the Fourier coefficients depend on the behaviour of f on the entire interval $(0, 1)$; one single discontinuity of f will for instance affect the size of all coefficients c_k and make the Fourier series converge slowly, thus making compression hard. It is also difficult or impossible to investigate local properties of f without summing the series and just look at the sequence of Fourier coefficients.

Wavelet expansions provide an alternative to Fourier series expansions. In such expansions, the basis functions are in general well-localized with respect to frequency and better localized with respect to space or time than the basis functions occurring in Fourier series expansions. A wavelet expansion is in fact a two-scale expansion. This feature makes it possible to study local phenomena of functions or signals with high accuracy.

For simplicity, we will study wavelet expansions of non-periodic functions on $\mathbf{R}$, where our main tool will be the Fourier transform on $L^2(\mathbf{R})$. It is also possible to do wavelet expansions of multidimensional signals, but in this exposition, we restrict ourselves to one-dimensional signals.

2. Preliminaries

The scalar field of any abstract Hilbert space H will be the complex numbers. The norm of a vector $x \in H$ is denoted by $\|x\|$ and the inner-product between two vectors $x, y \in H$ is denoted by (x, y). If $x_1, \ldots, x_n$ is a finite sequence in H of pairwise orthogonal vectors, meaning that $(x_k, x_l) = 0$ if $k \neq l$, then **Pythagoras' theorem** holds true:

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2. \tag{2.1}$$

This identity is proved simply by expanding the left-hand side using the properties of the inner-product. A sequence $(x_k)_{k \in \mathbf{Z}}$ is said to be **orthonormal** if the vectors are pairwise orthogonal and every vector has norm 1. Suppose that $(x_k)_{k \in \mathbf{Z}}$ is an orthonormal sequence in H and that $(a_k)_{k \in \mathbf{Z}}$ is a sequence of complex numbers. Using Pythagoras' theorem and the completeness of H, is easy to show that

$$x = \sum_{k \in \mathbf{Z}} a_k x_k \in H \quad \text{if and only if} \quad \sum_{k \in \mathbf{Z}} |a_k|^2 < \infty,$$

in which case

$$\|x\|^2 = \sum_{k \in \mathbf{Z}} |a_k|^2.$$

We will call the last identity **Parseval's theorem**. If $(x_k)_{k \in \mathbf{Z}}$ orthonormal sequence in H, then **Bessel's inequality**:

$$\sum_{k \in \mathbf{Z}} |(x, x_k)|^2 \leq \|x\|^2$$

holds for every vector $x \in H$. A consequence of Bessel's inequality is the fact that the series $\sum_{k \in \mathbf{Z}} (x, x_k) x_k$ is convergent in H for every vector $x \in H$. When this series equals x for any vector $x \in H$, then $(x_k)_{k \in \mathbf{Z}}$ is called an **orthonormal basis** for H.

The following are the most important examples of Hilbert spaces.

(i) Let $\ell^2(\mathbf{Z})$ denote the space of all sequences $a = (a_k)_{k\in\mathbf{Z}}$ of complex numbers that satisfy

$$\|a\|_2 = \Big(\sum_{k\in\mathbf{Z}} |a_k|^2\Big)^{1/2} < \infty.$$

This is a Hilbert space with the inner-product defined by

$$(a,b) = \sum_{k\in\mathbf{Z}} a_k\overline{b_k}, \quad a,b\in\ell^2(\mathbf{Z}).$$

(ii) Let $L^2(\mathbb{T})$ denote the space of all measurable, 2π-periodic functions f on $\mathbf{R}$ such that

$$\|f\|_2 = \Big(\int_0^{2\pi} |f(x)|^2\,dx\Big)^{1/2} < \infty.$$

This is a Hilbert space with the inner-product defined by

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx, \quad f,g\in L^2(\mathbb{T}).$$

It is well-known that if $f\in L^2(\mathbb{T})$, then

$$f(x) = \sum_{k\in\mathbf{Z}} \widehat{f}(k)e^{ikx},$$

where the series converges in $L^2(\mathbb{T})$ and the Fourier coefficients $\widehat{f}(k)$ are defined by

$$\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx, \quad k\in\mathbf{Z}.$$

The identity

$$\sum_{k\in\mathbf{Z}} \widehat{f}(k)\overline{\widehat{g}(k)} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx, \quad f,g\in L^2(\mathbb{T}),$$

is a consequence of Parseval's theorem.

(iii) Let $L^2(\mathbf{R})$ denote space of all measurable functions f on $\mathbf{R}$ such that

$$\|f\|_2 = \Big(\int_{\mathbf{R}} |f(x)|^2\,dx\Big)^{1/2} < \infty.$$

This is a Hilbert space with the inner-product defined by

$$(f,g) = \int_{\mathbf{R}} f(x)\overline{g(x)}\,dx, \quad f,g\in L^2(\mathbf{R}).$$

Let furthermore $L^1(\mathbf{R})$ denote the Banach spaces of all measurable functions f on $\mathbf{R}$ such that

$$\|f\|_1 = \int_{\mathbf{R}} |f(x)|\,dx < \infty.$$

The **Fourier transform** $\widehat{f}$ of a function $f\in L^1(\mathbf{R})$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}\,dx, \quad \xi\in\mathbf{R}.$$

If the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbf{R})$ is known and belongs to $L^1(\mathbf{R})$, then f can be reconstructed through the **inversion formula**:

$$f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{f}(\xi) e^{i\xi x}\, d\xi \quad \text{for a.e. } x \in \mathbf{R}.$$

We will use the well-known fact that the Fourier transform can be extended to $L^2(\mathbf{R})$ and denote the Fourier transform of $f \in L^2(\mathbf{R})$ by $\widehat{f}$. A consequence of the definition of the Fourier transform on $L^2(\mathbf{R})$ is the fact that if $f \in L^2(\mathbf{R})$, then $\widehat{f}$ also belongs to $L^2(\mathbf{R})$. One of the most useful results concerning the Fourier transform on $L^2(\mathbf{R})$ is **Plancherel's theorem**:

$$\int_{\mathbf{R}} f(x)\overline{g(x)}\, dx = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi, \quad f, g \in L^2(\mathbf{R}).$$

The inversion formula for $L^2(\mathbf{R})$ takes the form

$$f(x) = \frac{1}{2\pi} \operatorname{pv} \int_{\mathbf{R}} \widehat{f}(\xi) e^{i\xi x}\, d\xi \quad \text{for a.e. } x \in \mathbf{R},$$

where the principal value integral is defined by

$$\operatorname{pv} \int_{\mathbf{R}} \widehat{f}(\xi) e^{i\xi x}\, d\xi = \lim_{N \to \infty} \int_{|\xi| \le N} \widehat{f}(\xi) e^{i\xi x}\, d\xi.$$

Suppose that $\psi \in L^2(\mathbf{R})$. We shall use the notation

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad x \in \mathbf{R},$$

where $j, k \in \mathbf{Z}$. It is straight-forward to verify that

$$\|\psi_{j,k}\|_2 = \|\psi\|_2 \tag{2.2}$$

and

$$\widehat{\psi_{j,k}}(\xi) = 2^{-j/2} e^{-i2^{-j}k\xi} \widehat{\psi}(2^{-j}\xi), \quad \xi \in \mathbf{R}, \tag{2.3}$$

for all $j, k \in \mathbf{Z}$.

Let $\mathscr{S}$ the denote the Schwartz class, i.e., the collection of all infinitely differentiable functions f on $\mathbf{R}$ such that

$$\sup_{x \in \mathbf{R}} |t^j f^{(k)}(x)| < \infty \quad \text{for all integers } j, k \ge 0.$$

It is a well-known fact and easy to prove that the Fourier transform is a bijection between $\mathscr{S}$ and $\mathscr{S}$. In a proof in the last section, the space $\mathscr{S}'$ appears. This space is the dual space of $\mathscr{S}$, consisting of so-called tempered distributions.

We finally let χ_I denote the characteristic function for an interval $I \subset \mathbf{R}$ and use the notation

$$\operatorname{sinc} x = \frac{\sin x}{x}, \quad x \neq 0.$$

3. Wavelets

The following definition may be the most important in this text.

Definition 3.1. A function $\psi \in L^2(\mathbf{R})$ is called a **wavelet** if the system $(\psi_{j,k})_{j,k \in \mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$. This basis is then called a **wavelet basis** for $L^2(\mathbf{R})$.

At this stage, it is not at all clear that wavelets exist. Giving examples of wavelets and providing general tools for constructing wavelets with good properties with regards to localization and regularity are in fact the main objectives of this text.

Suppose that $(\psi_{j,k})_{j,k\in\mathbf{Z}}$ is a wavelet basis for $L^2(\mathbf{R})$. This means that every function $f \in L^2(\mathbf{R})$ can be written as

$$f(x) = \sum_{j,k\in\mathbf{Z}} c_{j,k}\psi_{j,k}(x),$$

where the series converges in $L^2(\mathbf{R})$ and the coefficients $c_{j,k}$ are given by

$$c_{j,k} = (f, \psi_{j,k}) = \int_{\mathbf{R}} f(x)\overline{\psi_{j,k}(x)}\,dx, \quad j,k \in \mathbf{Z}.$$

4. Multiresolution Analyses

We will in this exposition present a general framework for constructing wavelets. In this approach, the concept of a multiresolution analysis is central.

Definition 4.1. A **multiresolution analysis** is a sequence $(V_j)_{j\in\mathbf{Z}}$ of closed subspaces of $L^2(\mathbf{R})$ such that

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbf{Z}$;
(ii) $\bigcup_{j\in\mathbf{Z}} V_j$ is dense in $L^2(\mathbf{R})$;
(iii) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$;
(iv) $f(\,\cdot\,) \in V_j$ if and only if $f(2^{-j}\,\cdot\,) \in V_0$;
(v) $f(\,\cdot\,) \in V_0$ implies that $f(\,\cdot - k) \in V_0$ for every $k \in \mathbf{Z}$;
(vi) there exists a function $\phi \in V_0$ such that the system $(\phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal basis for V_0.

Axiom (i) is of course about **monotonicity** of the subspaces V_j. The third axiom is sometimes called the **separation axiom**. Axiom (iv) and (v) say that the spaces V_j should behave well under **scaling** and that the space V_0 is **translation invariant**, respectively. The function $\phi \in V_0$ in (vi) is called a **scaling function**. Let us mention that these axioms are not completely independent, but we will not go in to this isaue here.

Remark 4.2. Notice that (iv) and (vi) imply that the system $(\phi_{j,k})_{k\in\mathbf{Z}}$ is an orthonormal basis for V_j. Indeed, if $k \neq l$, then a change of variables shows that

$$\begin{aligned}(\phi_{j,k}, \phi_{j,l}) &= \int_{-\infty}^{\infty} 2^j\phi(2^jx-k)\overline{\phi(2^jx-l)}\,dx = \int_{-\infty}^{\infty} \phi(x-k)\overline{\phi(x-l)}\,dx \\ &= (\phi_{0,k}, \phi_{0,l}) = 0.\end{aligned}$$

Equation (2.2) also shows that $\|\phi_{j,k}\|_2 = \|\phi\|_2 = 1$ for every $k \in \mathbf{Z}$. Moreover, if $f \in V_j$, then $f(2^{-j}\,\cdot\,) \in V_0$, and therefore

$$f(2^{-j}x) = \sum_{k\in\mathbf{Z}} c_k\phi(x-k) \quad \text{in } L^2(\mathbf{R})$$

for some sequence $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$, so that

$$f(x) = \sum_{k\in\mathbf{Z}} c_k\phi(2^jx-k) = \sum_{k\in\mathbf{Z}} c_k'\phi_{j,k}(x) \quad \text{in } L^2(\mathbf{R}),$$

where $c_k' = 2^{-j/2}c_k,\ k \in \mathbf{Z}$.

Example 4.3. Our first example of a multiresolution analysis is the so-called **Haar system**.[1] Let V_j be the class of functions $f \in L^2(\mathbf{R})$ such that each function f is constant on every interval of the form $[2^{-j}n, 2^{-j}(n+1))$, $n \in \mathbf{Z}$. It is not so hard to show that each subspace V_j is a closed subset of $L^2(\mathbf{R})$. Out of the axioms in Definition 4.1, (i), (iv), and (v) are obvious.

Since step functions are dense in $L^2(\mathbf{R})$, (ii) holds.

To prove (iii), suppose that $f \in \bigcap_{j\in\mathbf{Z}} V_j$. Since $f \in V_j$ for every j, f is constant on every interval $[0, 2^{-j})$. Letting $j \to -\infty$, this shows that f is constant on $[0,\infty)$ and therefore that $f = 0$ on $[0,\infty)$ since $f \in L^2(\mathbf{R})$. In a similar manner, one shows that $f = 0$ on $(-\infty, 0]$, so $f = 0$.

Finally, take $\phi = \chi_{[0,1)}$, which will be our scaling function. Then the sequence $(\phi_{0,k})_{k\in\mathbf{Z}}$ is orthonormal since different functions in the sequence have disjoint supports. Also, if $f \in V_0$, then

$$f(x) = \sum_{k\in\mathbf{Z}} f(k)\phi(x-k) \quad \text{for every } x \in \mathbf{R}. \tag{4.1}$$

Since f is constant on every interval $[k, k+1)$, we have that

$$c_{0,k} = (f, \phi_{0,k}) = \int_{\mathbf{R}} f(x)\phi(x-k)\,dx = \int_k^{k+1} f(x)\phi(x-k)\,dx = f(k)$$

for every $k \in \mathbf{Z}$. It therefore follows from Bessel's inequality that

$$\Big\| f - \sum_{|k|\le N} c_{0,k}\phi_{0,k} \Big\|_2^2 \le \sum_{|k|\ge N+1} |f(k)|^2 \longrightarrow 0 \quad \text{as } N \to \infty$$

because

$$\sum_{k\in\mathbf{Z}} |f(k)|^2 = \sum_{k\in\mathbf{Z}} \int_k^{k+1} |f(x)|^2\,dx = \|f\|_2^2 < \infty,$$

which shows that (4.1) holds in $L^2(\mathbf{R})$, thus establishing (vi).

Example 4.4. Let V_j be the class of functions $f \in L^2(\mathbf{R})$ which are continuous and linear on every interval $[2^{-j}n, 2^{-j}(n+1))$, $n \in \mathbf{Z}$. The properties (i)–(v) in Definition 4.1 are verified more or less as in the previous example. We will come back to the choice of the scaling function and the proof of (vi) in the next section. This example relates to approximation with piecewise linear splines.

5. Orthonormal Sequences and Riesz Systems

In the present section, we study orthonormal sequences in Hilbert spaces. We also consider so-called Riesz systems, which are closely related to orthonormal sequences. We begin by showing that a system in a Hilbert space is orthonormal if and only if a version of Pythagoras theorem holds.

Lemma 5.1. *Suppose that H is a Hilbert space. Then a sequence $(x_k)_{k\in\mathbf{Z}}$ of vectors in H is orthonormal if and only if*

$$\Big\| \sum_{k\in\mathbf{Z}} a_k x_k \Big\|^2 = \sum_{k\in\mathbf{Z}} |a_k|^2 \tag{5.1}$$

[1] This system was introduced by the Hungarian mathematician Alfréd Haar in 1909 and provides an orthormal basis for $L^2(0,1)$.

for every sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ *such that* $a_k \neq 0$ *for not more than a finite number of elements in the sequence.*

Proof. The necessity part follows directly form Pythagoras' theorem (2.1).

To prove the converse, take $a_m = 1$ and $a_k = 0$ for $k \neq m$. Then (5.1) shows that $\|x_m\| = 1$. Then take $a_m = 1$, $a_n = -1$, and $a_k = 0$ for $k \neq m, n$ in (5.1):

$$2 = \|x_m - x_n\|^2 = 2 - 2\,\mathrm{Re}(x_m, x_n),$$

so $\mathrm{Re}(x_m, x_n) = 0$. Replacing x_m by ix_m, we obtain that $\mathrm{Im}(x_m, x_n) = 0$. This shows that $(x_m, x_n) = 0$. □

For a Riesz system, the left- and right-hand sides of (5.1) do not have to be equal, but should be equivalent in the following sense.

Definition 5.2. Suppose that H is a Hilbert space. A sequence $(x_k)_{k\in\mathbf{Z}}$ of vectors in H is called a **Riesz system** if there exist two constants $0 < C \le D$ such that

$$C\sum_{k\in\mathbf{Z}} |a_k|^2 \le \left\|\sum_{k\in\mathbf{Z}} a_k x_k\right\|^2 \le D\sum_{k\in\mathbf{Z}} |a_k|^2 \tag{5.2}$$

for every sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ such that $a_k \neq 0$ for not more than a finite number of elements in the sequence.

Remark 5.3.

(a) According to Lemma 5.1, any orthonormal sequence in a Hilbert space is a Riesz system with $C = D = 1$.

(b) Notice that every Riesz system is linearly independent. Indeed, suppose that $(x_k)_{k\in\mathbf{Z}}$ is a Riesz system and that

$$a_{k_1}x_{k_1} + \ldots + a_{k_n}x_{k_n} = 0$$

for some vectors in the sequence and some complex coefficients. Then (5.2) shows that $|a_{k_1}|^2 + \ldots + |a_{k_n}|^2 = 0$, so $a_{k_1} = \ldots = a_{k_n} = 0$.

Example 5.4. Take $H = L^2(\mathbf{R})$ and let ϕ be the "tent function" defined by

$$\phi(x) = (1 - |x|)\chi_{[-1,1]}(x),\ \ x \in \mathbf{R}.$$

If $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ is a sequence as in Definition 5.2, then

$$\begin{aligned}\left\|\sum_{k\in\mathbf{Z}} a_k\phi_{0,k}\right\|_2^2 &= \sum_{n\in\mathbf{Z}}\int_n^{n+1} |a_n(n+1-x) + a_{n+1}(x-n)|^2\,dx\\ &= \frac{1}{3}\sum_{n\in\mathbf{Z}}(|a_n|^2 + |a_{n+1}|^2 + \mathrm{Re}(a_n\overline{a_{n+1}})).\end{aligned}$$

Using the inequality

$$2|\,\mathrm{Re}(a_n\overline{a_{n+1}})| \le 2|a_n||a_{n+1}| \le |a_n|^2 + |a_{n+1}|^2,$$

we obtain that

$$\left\|\sum_{k\in\mathbf{Z}} a_k x_k\right\|_2^2 \le \frac{1}{2}\sum_{n\in\mathbf{Z}}(|a_n|^2 + |a_{n+1}|^2) = \sum_{n\in\mathbf{Z}}|a_n|^2$$

and

$$\left\|\sum_{k\in\mathbf{Z}} a_k x_k\right\|_2^2 \ge \frac{1}{6}\sum_{n\in\mathbf{Z}}(|a_n|^2 + |a_{n+1}|^2) = \frac{1}{3}\sum_{n\in\mathbf{Z}}|a_n|^2.$$

This shows that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system with $C=\frac{1}{3}$ and $D=1$.

6. Orthonormal and Riesz Systems Generated by Translates

We next consider systems in $L^2(\mathbf{R})$, generated by all integer translates of a function $\phi\in L^2(\mathbf{R})$, and prove an equivalent condition for such a system to be a Riesz system. This condition is given in terms of the Fourier transform of ϕ.

Theorem 6.1. *Suppose that $\phi\in L^2(\mathbf{R})$. Then $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system in $L^2(\mathbf{R})$ with constants $0<C\le D$ if and only if*

$$C\le\sum_{k\in\mathbf{Z}}|\widehat{\phi}(\xi+2k\pi)|^2\le D\quad\textit{for a.e. }\xi\in\mathbf{R}.\tag{6.1}$$

Remark 6.2. Notice that

$$\int_0^{2\pi}\sum_{k\in\mathbf{Z}}|\widehat{\phi}(\xi+2k\pi)|^2\,d\xi=\sum_{k\in\mathbf{Z}}\int_0^{2\pi}|\widehat{\phi}(\xi+2k\pi)|^2\,d\xi=\sum_{k\in\mathbf{Z}}\int_{2k\pi}^{2(k+1)\pi}|\widehat{\phi}(\xi)|^2\,d\xi$$
$$=\int_{\mathbf{R}}|\widehat{\phi}(\xi)|^2\,d\xi<\infty.$$

This shows that the series in (6.1) is defined and finite a.e. on $(0,2\pi)$ and hence a.e. on $\mathbf{R}$ since it is periodic with period 2π. The series is the so-called **periodization** of the square of the absolute value of $\widehat{\phi}$, which therefore belongs to $L^1(\mathbb{T})$. Under the assumption (6.1), the periodization belongs to $L^2(\mathbb{T})$.

Before proving this theorem, let us mention the following important corollary.

Corollary 6.3. *Suppose that $\phi\in L^2(\mathbf{R})$. Then $(\phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal sequence in $L^2(\mathbf{R})$ if and only if*

$$\sum_{k\in\mathbf{Z}}|\widehat{\phi}(\xi+2k\pi)|^2=1\quad\textit{for a.e. }\xi\in\mathbf{R}.$$

Proof of Theorem 6.1. Suppose that $(a_k)_{k\in\mathbf{Z}}\in\ell^2(\mathbf{Z})$ and $a_k\ne0$ for not more than a finite number of elements in the sequence. Put

$$\Phi(x)=\sum_{k\in\mathbf{Z}}a_k\phi(x-k),\quad x\in\mathbf{R}.$$

The Fourier transform of Φ is

$$\widehat{\Phi}(\xi)=m(\xi)\widehat{\phi}(\xi),\quad\xi\in\mathbf{R},$$

where m is the trigonometric polynomial given by

$$m(\xi)=\sum_{k\in\mathbf{Z}}a_ke^{-ik\xi},\quad\xi\in\mathbf{R}.$$

Plancherel's theorem now shows that

$$2\pi\|\Phi\|_2^2=\int_{\mathbf{R}}|\widehat{\Phi}(\xi)|^2\,d\xi=\int_{\mathbf{R}}|m(\xi)|^2|\widehat{\phi}(\xi)|^2\,d\xi=\sum_{l\in\mathbf{Z}}\int_{2l\pi}^{2(l+1)\pi}|m(\xi)|^2|\widehat{\phi}(\xi)|^2\,d\xi$$
$$=\sum_{l\in\mathbf{Z}}\int_0^{2\pi}|m(\xi)|^2|\widehat{\phi}(\xi+2l\pi)|^2\,d\xi=\int_0^{2\pi}|m(\xi)|^2\Big(\sum_{l\in\mathbf{Z}}|\widehat{\phi}(\xi+2l\pi)|^2\Big)\,d\xi,$$

that is

$$\Big\|\sum_{k\in\mathbf{Z}} a_k\phi_{0,k}\Big\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |m(\xi)|^2\Big(\sum_{l\in\mathbf{Z}} |\widehat{\phi}(\xi+2l\pi)|^2\Big)\,d\xi. \tag{6.2}$$

If we suppose that (6.1) holds, then since

$$\frac{1}{2\pi}\int_0^{2\pi} |m(\xi)|^2\,d\xi = \sum_{k\in\mathbf{Z}} |a_k|^2$$

according to Parseval's Theorem, (6.2) immediately shows that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system.

Conversely, assuming that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system, then (6.2) implies that

$$C\int_0^{2\pi} |m(\xi)|^2\,d\xi \le \int_0^{2\pi} |m(\xi)|^2\Big(\sum_{l\in\mathbf{Z}} |\widehat{\phi}(\xi+2l\pi)|^2\Big)\,d\xi \le D\int_0^{2\pi} |m(\xi)|^2\,d\xi$$

for every trigonometric polynomial m. Replacing m by a sequence of trigonometric polynomials that converges boundedly to the characteristic function of an interval $[a,b]\subset(0,2\pi)$, we obtain that

$$C \le \frac{1}{b-a}\int_a^b \Big(\sum_{l\in\mathbf{Z}} |\widehat{\phi}(\xi+2l\pi)|^2\Big)\,d\xi \le D.$$

If we finally let (a,b) shrink to a point $\xi_0\in(0,2\pi)$, then Lebesgue's differentiation theorem shows that the inequality in (6.1) holds a.e. □

Example 6.4. Consider again the Haar system in Example 4.3 with scaling function $\phi=\chi_{[0,1)}$. The Fourier transform of ϕ is

$$\widehat{\phi}(\xi) = e^{-i\xi/2}\frac{\sin\xi/2}{\xi/2}, \quad \xi\neq 0.$$

Since $(\phi_{0,k})_{k\in\mathbf{Z}}$ is orthonormal in $L^2(\mathbf{R})$, Corollary 6.3 shows that

$$1 = \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)|^2 = 4\sum_{k\in\mathbf{Z}} \frac{\sin^2\xi/2}{(\xi+2k\pi)^2}$$

for a.e. $\xi\in\mathbf{R}$. Since the right-hand side in this identity is uniformly convergent on any closed interval, which does not contain multiples of 2π, and therefore represents a continuous function on such an interval, the identity holds for every $\xi\in\mathbf{R}$, which is not an multiple of 2π. This shows that

$$\frac{1}{4\sin^2\xi/2} = \sum_{k\in\mathbf{Z}} \frac{1}{(\xi+2k\pi)^2}, \quad \xi\notin 2\pi\mathbf{Z}, \tag{6.3}$$

which is the expansion of the function $(4\sin^2\xi/2)^{-1}$ into partial fractions.

Example 6.5. Take $f(x)=\chi_{(-\pi,\pi)}(x),\ x\in\mathbf{R}$. Then

$$\widehat{f}(\xi) = 2\frac{\sin\pi\xi}{\xi} = 2\pi\,\mathrm{sinc}\,\pi\xi, \quad \xi\in\mathbf{R}.$$

The inversion formula therefore shows that if $\phi(x)=\mathrm{sinc}\,\pi x,\ x\in\mathbf{R}$, then

$$\widehat{\phi}(\xi) = \chi_{(-\pi,\pi)}(\xi), \quad \xi\in\mathbf{R}.$$

Notice that

$$\sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)|^2 = \sum_{k\in\mathbf{Z}} \chi_{(-\pi,\pi)}(\xi+2k\pi) = 1$$

for a.e. $\xi \in \mathbf{R}$. This implies that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in $L^2(\mathbf{R})$. We will call this system the **Shannon system** since it is connected with Shannon's sampling theorem.

The following corollary gives a substitute for Parseval's theorem for Riesz systems of the form $(\phi_{0,k})_{k\in\mathbf{Z}}$, where $\phi \in L^2(\mathbf{R})$. The equality in Parseval's theorem is here replaced by two inequalities.

Corollary 6.6. *Suppose that $\phi \in L^2(\mathbf{R})$ and that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system in $L^2(\mathbf{R})$ with constants $0 < C \le D$. Then*

$$C\sum_{k\in\mathbf{Z}} |a_k|^2 \le \Big\|\sum_{k\in\mathbf{Z}} a_k\phi_{0,k}\Big\|_2^2 \le D\sum_{k\in\mathbf{Z}} |a_k|^2 \tag{6.4}$$

for any sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$.

Proof. Let $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$. Since $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system,

$$C\sum_{|k|\le N} |a_k|^2 \le \Big\|\sum_{|k|\le N} a_k\phi_{0,k}\Big\|_2^2 \le D\sum_{|k|\le N} |a_k|^2 \tag{6.5}$$

for any integer $N \ge 0$. As in the proof of Theorem 6.1 (see in particular (6.2)), we have that

$$\Big\|\sum_{|k|\le N} a_k\phi_{0,k}\Big\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |m_N(\xi)|^2\Big(\sum_{l\in\mathbf{Z}} |\widehat{\phi}(\xi+2l\pi)|^2\Big)\,d\xi,$$

where

$$m_N(\xi) = \sum_{|k|\le N} a_k e^{-ik\xi}, \quad \xi \in \mathbf{R}.$$

According to Parseval's theorem, m_N tends to the function $m \in L^2(\mathbb{T})$, given by

$$m(\xi) = \sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}, \quad \xi \in \mathbf{R},$$

in $L^2(\mathbb{T})$ as $N \to \infty$. It therefore follows from (6.1) that the middle member in (6.5) tends to the middle member in (6.4) as $N \to \infty$. The left- and right-hand side in (6.5) obviously tend to the left- and right-hand side in (6.4), respectively. □

The next lemma describes the closed linear span of a Riesz system. The closed linear span of a subset of $L^2(\mathbf{R})$ is of course the closure in $L^2(\mathbf{R})$ of the linear span of the subset.

Lemma 6.7. *Suppose that $\phi \in L^2(\mathbf{R})$ and $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system in $L^2(\mathbf{R})$. Then a function $f \in L^2(\mathbf{R})$ belongs to the closed linear span of $(\phi_{0,k})_{k\in\mathbf{Z}}$ if and only if*

$$f(x) = \sum_{k\in\mathbf{Z}} a_k\phi_{0,k}(x) \tag{6.6}$$

in $L^2(\mathbf{R})$ for some sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$.

Proof. The sufficiency part is straightforward since any partial sum to the series in (6.6) belongs to the linear span of $(\phi_{0,k})_{k\in\mathbf{Z}}$ and converges to f in $L^2(\mathbb{T})$.

Now suppose that f is in the closed linear span of $(\phi_{0,k})_{k\in\mathbf{Z}}$. Then there exists a sequence $(f_n)_{n=1}^{\infty}$, where each function f_n belongs to the linear span of $(\phi_{0,k})_{k\in\mathbf{Z}}$, i.e.,

$$f_n(x) = \sum_{k\in\mathbf{Z}} a_k^{(n)} \phi_{0,k}(x), \quad x \in \mathbf{R},$$

and $a_k^{(n)} \neq 0$ for not more than a finite number of k, such that $f_n \to f$ in $L^2(\mathbf{R})$. Put $a_n = (a_k^{(n)})_{k\in\mathbf{Z}}$ for $n = 1, 2, \ldots$. According to (5.2),

$$C\|a_m - a_n\|_2^2 \leq \|f_m - f_n\|_2^2,$$

which shows that a_n converges to some sequence $a = (a_k)_{k\in\mathbf{Z}}$ in $\ell^2(\mathbf{Z})$. Put

$$g_K(x) = \sum_{|k|\leq K} a_k \phi_{0,k}(x), \quad x \in \mathbf{R},$$

for $K = 0, 1, \ldots$. Then, given $\varepsilon > 0$, choose n so large that

$$\|f - f_n\|_2 < \varepsilon \quad \text{and} \quad \|a - a_n\|_2 < \varepsilon.$$

Now, if K is so large that $a_k^{(n)} = 0$ for $|k| \geq K + 1$, then

$$\|f - g_K\|_2 \leq \|f - f_n\|_2 + \|f_n - g_K\|_2 < \varepsilon + \sqrt{D}\|a_n - a\|_2 < (1 + \sqrt{D})\varepsilon.$$

This proves that (6.6) holds in $L^2(\mathbf{R})$. □

Remark 6.8. Suppose that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system in $L^2(\mathbf{R})$. Then $(\phi_{j,k})_{k\in\mathbf{Z}}$ is also a Riesz system for every $j \in \mathbf{Z}$ since

$$\begin{aligned}\Big\|\sum_{k\in\mathbf{Z}} a_k \phi_{j,k}\Big\|_2^2 &= \int_{\mathbf{R}} \Big|\sum_{k\in\mathbf{Z}} a_k 2^{j/2}\phi(2^j x - k)\Big|^2 dx = \int_{\mathbf{R}} \Big|\sum_{k\in\mathbf{Z}} a_k \phi(x - k)\Big|^2 dx \\ &= \Big\|\sum_{k\in\mathbf{Z}} a_k \phi_{0,k}\Big\|_2^2\end{aligned}$$

for any sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ with not more than a finite number of non-zero of elements. The lemma therefore shows that the closed linear span of $(\phi_{j,k})_{k\in\mathbf{Z}}$ consists of all functions $f \in \ell^2(\mathbf{Z})$ of the form

$$f(x) = \sum_{k\in\mathbf{Z}} a_k \phi_{j,k}(x),$$

where $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ and the series converges in $L^2(\mathbf{R})$.

The next theorem gives a way to orthogonalize a Riesz system. Starting with a Riesz system, one obtains an orthonormal system with the same closed linear span as the original system.

Theorem 6.9. *Suppose that $\phi \in L^2(\mathbf{R})$ and $(\phi_{0,k})_{k\in\mathbf{Z}}$ is a Riesz system in $L^2(\mathbf{R})$.*

(a) *There exists a sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ and a function $\Phi \in L^2(\mathbf{R})$ of the form*

$$\Phi(x) = \sum_{k\in\mathbf{Z}} a_k \phi_{0,k}(x), \quad x \in \mathbf{R}, \tag{6.7}$$

such that $(\Phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in $L^2(\mathbf{R})$.

(b) *The closed linear span of the system* $(\phi_{0,k})_{k\in\mathbf{Z}}$ *equals the closed linear span of* $(\Phi_{0,k})_{k\in\mathbf{Z}}$.

Proof. (a) Put

$$\widehat{\Phi}(\xi)=\frac{\widehat{\phi}(\xi)}{\left(\sum_{l\in\mathbf{Z}}|\widehat{\phi}(\xi+2l\pi)|^2\right)^{1/2}},\quad \xi\in\mathbf{R}. \tag{6.8}$$

According to Theorem 6.1 (see (6.1)), the denominator in this quotient is bounded from away from 0 from below and from above, which means that $\widehat{\Phi}$ is defined and that $\widehat{\Phi}$ belongs to $L^2(\mathbf{R})$. It follows that the inverse Fourier transform of Φ also belongs to $L^2(\mathbf{R})$. Since the denominator has period 2π, we also have that

$$\sum_{k\in\mathbf{Z}}|\widehat{\Phi}(\xi+2k\pi)|^2=1$$

for a.e. $\xi\in\mathbf{R}$. Corollary 6.3 now shows that $(\Phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in $L^2(\mathbf{R})$. Let m denote the function

$$m(\xi)=\left(\sum_{l\in\mathbf{Z}}|\widehat{\phi}(\xi+2l\pi)|^2\right)^{-1/2},\quad \xi\in\mathbf{R}.$$

Then m has period 2π and is bounded and therefore belongs to $L^2(\mathbb{T})$, so

$$m(\xi)=\sum_{k\in\mathbf{Z}}a_ke^{-ik\xi}$$

for some sequence $(a_k)_{k\in\mathbf{Z}}\in\ell^2(\mathbf{Z})$. Inverting (6.8), we therefore obtain (6.7).

(b) Consider the equation

$$\sum_{k\in\mathbf{Z}}b_k\phi_{0,k}=\sum_{k\in\mathbf{Z}}c_k\Phi_{0,k}, \tag{6.9}$$

where either $(b_k)_{k\in\mathbf{Z}}\in\ell^2(\mathbf{Z})$ is unknown or $(c_k)_{k\in\mathbf{Z}}\in\ell^2(\mathbf{Z})$ is unknown. Taking Fourier transforms, we see that this equation is equivalent to

$$B(\xi)\widehat{\phi}(\xi)=C(\xi)\widehat{\Phi}(\xi)=C(\xi)m(\xi)\widehat{\phi}(\xi),\quad \xi\in\mathbf{R},$$

where

$$B(\xi)=\sum_{k\in\mathbf{Z}}b_ke^{-ik\xi}\quad\text{and}\quad C(\xi)=\sum_{k\in\mathbf{Z}}c_ke^{-ik\xi}.$$

Suppose first that $(b_k)_{k\in\mathbf{Z}}$ is known. We then get a solution to equation (6.9) by taking $C=B/m\in L^2(\mathbb{T})$ and choosing $(c_{-k})_{k\in\mathbf{Z}}$ as the Fourier coefficients of C. This shows that the closed linear span of $(\Phi_{0,k})_{k\in\mathbf{Z}}$ is a subset of the closed linear span of $(\phi_{0,k})_{k\in\mathbf{Z}}$. For the opposite inclusion suppose that $(c_k)_{k\in\mathbf{Z}}$ is known. Then choose $B=mC$ and $(b_{-k})_{k\in\mathbf{Z}}$ as the Fourier coefficients of C. □

Example 6.10. Let us return to Example 5.4, where

$$\phi(x)=(1-|x|)\chi_{[-1,1]}(x),\ x\in\mathbf{R}.$$

A simple calculation shows that

$$\widehat{\phi}(\xi)=4\frac{\sin^2\xi/2}{\xi^2},\quad \xi\neq 0.$$

It follows that

$$\sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)|^2 = 16\sin^4\xi/2 \sum_{k\in\mathbf{Z}} \frac{1}{(\xi+2k\pi)^4}, \quad \xi \notin 2\pi\mathbf{Z}.$$

Differentiating (6.3) twice, we obtain that

$$\sum_{k\in\mathbf{Z}} \frac{1}{(\xi+2k\pi)^4} = \frac{1+\cos^2\xi/2}{3\sin^4\xi/2},$$

so

$$\sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)|^2 = \frac{16}{3}(1+\cos^2\xi/2).$$

Applying (6.8), we therefore see that the function $\widehat{\Phi}$ therefore is given by

$$\widehat{\Phi}(\xi) = \frac{\sqrt{3}\sin^2\xi/2}{\xi^2\sqrt{1+\cos^2\xi/2}} = \frac{\sqrt{3}}{4\sqrt{1+\cos^2\xi/2}}\widehat{\phi}(\xi).$$

It is at least in principle possible to find Φ by inverting this formula by expanding the function

$$m(\xi) = \frac{\sqrt{3}}{4\sqrt{1+\cos^2\xi/2}}, \quad \xi \in \mathbf{R},$$

into a Fourier series with period 2π.

7. The Scaling Equation and the Structure Constants

Let $\phi \in L^2(\mathbf{R})$ be the scaling function for a multiresolution analysis $(V_j)_{j\in\mathbf{Z}}$. Then, for every function $f \in V_1$, there exists a sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ such that

$$f(x) = \sum_{k\in\mathbf{Z}} a_k \phi_{1,k}(x) \tag{7.1}$$

in $L^2(\mathbf{R})$. The coefficients a_k are of course given by inner products:

$$a_k = (f, \phi_{1,k}) = 2^{1/2} \int_{\mathbf{R}} f(x)\overline{\phi(2x-k)}\,dx, \quad k \in \mathbf{Z}. \tag{7.2}$$

In particular, since $\phi \in V_0 \subset V_1$,

$$\phi(x) = \sum_{k\in\mathbf{Z}} c_k \phi_{1,k}(x) \tag{7.3}$$

in $L^2(\mathbf{R})$. The equation (7.3) is called the **scaling equation** and the numbers c_k are known as the **structure constants**.

Example 7.1. The scaling function for the Haar system is $\phi = \chi_{[0,1)}$. Since

$$\phi(x) = \phi(2x) + \phi(2x-1) \quad \text{for every } x \in \mathbf{R},$$

the scaling equation is

$$\phi(x) = \tfrac{1}{\sqrt{2}}\phi_{1,0}(x) + \tfrac{1}{\sqrt{2}}\phi_{1,1}(x)$$

and the structure constants are $c_0 = c_1 = \frac{1}{\sqrt{2}}$ and $c_k = 0$ for $k \neq 0, 1$.

We leave the proof of the following proposition as an exercise to the reader.

Proposition 7.2. *Suppose that $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$ are the structure constants for a multiresolution analysis. Then*

$$\sum_{k\in\mathbf{Z}} c_k\overline{c_{2m+k}} = \begin{cases} 1 & \text{if} \quad m=0 \\ 0 & \text{if} \quad m\neq 0 \end{cases}.$$

Theorem 7.3. *Suppose that $\phi \in L^2(\mathbf{R})$ is the scaling function for a multiresolution analysis $(V_j)_{j\in\mathbf{Z}}$. Then, for every $f \in V_1$, there exists a function $m_f \in L^2(\mathbb{T})$ such that*

$$\widehat{f}(\xi) = m_f(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2}) \quad \text{for a.e. every } \xi \in \mathbf{R}. \tag{7.4}$$

The function m_f is given by

$$m_f(\xi) = \tfrac{\sqrt{2}}{2}\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}, \quad \xi \in \mathbf{R}, \tag{7.5}$$

where the coefficients a_k are given by (7.2).

The function m_f, that appears in (7.4), is called the **filter** associated with f and (7.4) the **filter identity**.

Proof. According to (7.1),

$$f(x) = \sum_{k\in\mathbf{Z}} a_k\phi_{1,k}(x)$$

in $L^2(\mathbf{R})$. Taking Fourier transforms of both sides, using the fact that the Fourier transform of $\phi_{1,k}$ is

$$\begin{aligned} 2^{1/2}\int_{\mathbf{R}} \phi(2x-k)e^{-ix\xi}\,dx &= 2^{-1/2}\int_{\mathbf{R}} \phi(y)e^{-i(y+k)\xi/2}\,dx \\ &= 2^{-1/2}e^{-ik\xi/2}\widehat{\phi}(\tfrac{\xi}{2}), \end{aligned} \tag{7.6}$$

we see that

$$\widehat{f}(\xi) = \tfrac{\sqrt{2}}{2}\Big(\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi/2}\Big)\widehat{\phi}(\tfrac{\xi}{2}).$$

If we finally let

$$m_f(\xi) = \tfrac{\sqrt{2}}{2}\sum_{k\in\mathbf{Z}} a_k e^{-ik\xi}, \quad \xi \in \mathbf{R},$$

then (7.4) holds and $m_f \in L^2(\mathbb{T})$ since $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{R})$. □

Example 7.4. As we saw in Example 7.1, the scaling equation for the Haar system is

$$\phi(x) = \tfrac{1}{\sqrt{2}}\phi_{1,0}(x) + \tfrac{1}{\sqrt{2}}\phi_{1,1}(x), \quad x \in \mathbf{R}.$$

We also have that (see (7.6)

$$\widehat{\phi_{0,1}}(\xi) = 2^{-1/2}\widehat{\phi}(\tfrac{\xi}{2}) \quad \text{and} \quad \widehat{\phi_{1,1}}(\xi) = 2^{-1/2}e^{-i\xi/2}\widehat{\phi}(\tfrac{\xi}{2})$$

for $\xi \in \mathbf{R}$, so the filter identity is

$$\widehat{\phi}(\xi) = m_\phi(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2}), \quad \text{where} \quad m_\phi(\xi) = \frac{1}{2}(1+e^{-i\xi}),\ \xi \in \mathbf{R}.$$

Example 7.5. For the Shannon system in Example 6.5, the scaling function is $\phi(x) = \operatorname{sinc} \pi x,\ x \in \mathbf{R}$, with

$$\widehat{\phi}(\xi) = \chi_{(-\pi,\pi)}(\xi), \quad \xi \in \mathbf{R}.$$

The filter identity therefore takes the form

$$\chi_{(-\pi,\pi)}(\xi) = m_\phi(\tfrac{\xi}{2})\chi_{(-\pi,\pi)}(\tfrac{\xi}{2}) = m_\phi(\tfrac{\xi}{2})\chi_{(-2\pi,2\pi)}(\xi), \quad \xi \in \mathbf{R}.$$

This identity holds if we choose $m_\phi = \chi_{(-\pi/2,\pi/2)}$. Since the Fourier coefficients of m_ϕ are

$$\widehat{m_\phi}(k) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{ik\xi}\, d\xi = \begin{cases} \dfrac{\sin k\pi/2}{k\pi} & \text{for} \quad k \neq 0 \\ \dfrac{1}{2} & \text{for} \quad k = 0 \end{cases},$$

the structure constants are

$$c_k = \frac{\sqrt{2}\sin k\pi/2}{k\pi},\ k \neq 0, \quad \text{and} \quad c_0 = \frac{\sqrt{2}}{2}.$$

Example 7.6.

8. Generating a Multiresolution Analysis

Theorem 8.1. *Suppose that $\phi \in L^2(\mathbf{R})$ and*

(i) $(\phi_{0,k})_{k\in\mathbf{Z}}$ *is an orthonormal system in* $L^2(\mathbf{R})$;

(ii) ϕ *satisfies a scaling equation*:

$$\phi(x) = \sum_{k\in\mathbf{Z}} c_k \phi_{1,k}(x)$$

in $L^2(\mathbf{R})$ for some sequence $(c_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$;

(iii) $\widehat{\phi}$ *is continuous at* 0 *with* $|\widehat{\phi}(0)| = 1$.

For $j \in \mathbf{Z}$, let V_j denote the closed linear span of $(\phi_{j,k})_{k\in\mathbf{Z}}$. Then $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$.

Remark 8.2. According to Remark 6.8, V_j consists of all functions $f \in L^2(\mathbf{R})$ such that

$$f(x) = \sum_{k\in\mathbf{Z}} a_k \phi_{j,k}(x)$$

in $L^2(\mathbf{R})$ for some sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$.

Proof of Theorem 8.1. We need to verify all six axioms in Definition 4.1. Suppose first that $f \in V_j$, i.e.,

$$f(x) = \sum_{k\in\mathbf{Z}} a_k \phi_{j,k}(x) = \sum_{k\in\mathbf{Z}} (2^{j/2}a_k)\phi(2^j x - k)$$

in $L^2(\mathbf{R})$ for some sequence $(a_k)_{k\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$. The scaling equation now shows that

$$\phi(2^j x - k) = \sum_{l\in\mathbf{Z}} c_l \phi_{1,l}(2^j x - k) = \sum_{l\in\mathbf{Z}} (2^{1/2}c_l)\phi(2^{j+1}x - 2k - l).$$

Inserting the last identity in the previous, we see that

$$f(x) = \sum_{k,l\in\mathbf{Z}} (2^{(j+1)/2}a_k c_l)\phi(2^{j+1}x - 2k - l),$$

which after renumbering is seen to be an element of V_{j+1}. Thus, $V_j \subset V_{j+1}$.

We will next check that $f(\,\cdot\,) \in V_j$ if and only if $f(2^{-j}\,\cdot\,) \in V_0$. If $f \in V_j$, then

$$f(2^{-j}x) = \sum_{k\in\mathbf{Z}} a_k\phi_{j,k}(2^{-j}x) = \sum_{k\in\mathbf{Z}} (2^{j/2}a_k)\phi_{0,k}(x),$$

which shows that $f(2^{-j}\,\cdot\,) \in V_0$. Suppose conversely that $f(2^{-j}\,\cdot\,) \in V_0$, i.e.,

$$f(2^{-j}x) = \sum_{k\in\mathbf{Z}} a_k\phi_{0,k}(x).$$

Then

$$f(x) = \sum_{k\in\mathbf{Z}} a_k\phi(2^jx-k) = \sum_{k\in\mathbf{Z}} (2^{-j/2}a_k)\phi_{j,k}(x),$$

showing that $f \in V_j$.

The space V_0 is also invariant under translations, since if

$$f(x) = \sum_{k\in\mathbf{Z}} \phi(x-k), \quad \text{then} \quad f(x-l) = \sum_{k\in\mathbf{Z}} \phi(x-(k+l)).$$

The remaining parts of the proof are consequences of the following two lemmas. $\square$

Let P_jf denote the orthogonal projection of a function $f \in L^2(\mathbf{R})$ on V_j for $j \in \mathbf{Z}$, i.e.,

$$P_jf = \sum_{k\in\mathbf{Z}} (f,\phi_{j,k})\phi_{j,k}.$$

Lemma 8.3. *Suppose that $f \in L^2(\mathbf{R})$. Then $P_jf \to 0$ in $L^2(\mathbf{R})$ as $j \to -\infty$.*

This lemma implies that $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$. Indeed, if $f \in V_j$ for every $j \in \mathbf{Z}$, then $P_jf = f$, so that $f = \lim_{j\to-\infty} P_jf = 0$.

Proof of Lemma 8.3. Put $f_R = f\chi_{[-R,R]}$, $R > 0$. Since

$$\|P_jf\|_2 \le \|P_j(f-f_R)\|_2 + \|P_jf_R\|_2 \le \|f-f_R\|_2 + \|P_jf_R\|_2$$

and the first term in the right-hand side of this inequality tends to 0 as $R \to \infty$, we can assume that f has compact support, i.e., $\operatorname{supp} f \subset [-R,R]$ for some $R > 0$. Then, according to Parseval's identity and Hölder's inequality,

$$\begin{aligned}
\|P_jf\|_2^2 = \sum_{k\in\mathbf{Z}} |(f,\phi_{j,k})|^2 &\le \sum_{k\in\mathbf{Z}} \left(\int_{-R}^{R} |f(x)||\phi_{j,k}(x)|\,dx\right)^2 \\
&\le \sum_{k\in\mathbf{Z}} \int_{-R}^{R} |f(x)|^2\,dx \int_{-R}^{R} 2^j|\phi(2^jx-k)|^2\,dx \\
&= \|f\|_2^2 \int_{-2^jR-k}^{2^jR-k} |\phi(y)|^2\,dy.
\end{aligned}$$

Notice finally that the last integral tends to 0 as $j \to -\infty$ due to the Dominated Convergence Theorem. $\square$

Lemma 8.4. *Suppose that $f \in L^2(\mathbf{R})$. Then $P_jf \to f$ in $L^2(\mathbf{R})$ as $j \to \infty$.*

This lemma implies that $\bigcup_{j\in\mathbf{Z}} V_j$ is dense in $L^2(\mathbf{R})$.

Proof of Lemma 8.4. Let $\widehat{f_n} \in C_c^\infty(\mathbf{R})$ such that $\widehat{f_n} \to \widehat{f}$ in $L^2(\mathbf{R})$. Then $f \in \mathscr{S}$ and $f_n \to f$ in $L^2(\mathbf{R})$ according to Plancherel's theorem. Moreover, since

$$\begin{aligned}\|f - P_j f\|_2 &\le \|f - f_n\|_2 + \|f_n - P_j f_n\|_2 + \|P_j(f_n - f)\|_2 \\ &\le 2\|f - f_n\|_2 + \|f_n - P_j f_n\|_2,\end{aligned}$$

we see that it suffices to prove the lemma in the case when $\widehat{f}$ is bounded and has compact support. Suppose therefore that $\widehat{f}$ is bounded and $\operatorname{supp} \widehat{f} \subset [-R, R]$ for some $R > 0$. Because

$$\|f - P_j f\|_2^2 = \|f\|_2^2 - \|P_j f\|_2^2$$

according to Pythagoras' theorem, we only need to show that $\|P_j f\|_2 \to \|f\|_2$ as $j \to \infty$. Suppose that j is so large that $2^{-j} R \le \pi$. Applying first Parseval's and then Plancherel's theorems, we obtain that

$$\|P_j f\|_2^2 = \sum_{k \in \mathbf{Z}} |(f, \phi_{j,k})|^2 = \frac{1}{(2\pi)^2} \sum_{k \in \mathbf{Z}} \left| \int_{-R}^{R} \widehat{f}(\xi) \overline{\widehat{\phi_{j,k}}(\xi)}\, d\xi \right|^2.$$

We then use (2.3) for the Fourier transform of $\phi_{j,k}$ and change of variables $2^{-j}\xi = \omega$:

$$\begin{aligned}\int_{-R}^{R} \widehat{f}(\xi) \overline{\widehat{\phi_{j,k}}(\xi)}\, d\xi &= 2^{-j/2} \int_{-R}^{R} \widehat{f}(\xi) e^{i2^{-j}k\xi} \overline{\widehat{\phi}(2^{-j}\xi)}\, d\xi \\ &= 2^{j/2} \int_{-2^{-j}R}^{2^{-j}R} \widehat{f}(2^j\omega) \overline{\widehat{\phi}(\omega)} e^{ik\omega}\, d\omega \\ &= 2^{j/2} \int_{-\pi}^{\pi} \widehat{f}(2^j\omega) \overline{\widehat{\phi}(\omega)} e^{ik\omega}\, d\omega.\end{aligned}$$

Notice that the assumption on f implies that the integrand in the last integral belongs to $L^2(-\pi, \pi)$. Parseval's theorem therefore shows that

$$\begin{aligned}\|P_j f\|_2^2 &= \frac{2^j}{(2\pi)^2} \sum_{k \in \mathbf{Z}} \left| \int_{-\pi}^{\pi} \widehat{f}(2^j\omega) \overline{\widehat{\phi}(\omega)} e^{ik\omega}\, d\omega \right|^2 = \frac{2^j}{2\pi} \int_{-\pi}^{\pi} |\widehat{f}(2^j\omega)|^2 |\widehat{\phi}(\omega)|^2\, d\omega \\ &= \frac{1}{2\pi} \int_{-R}^{R} |\widehat{f}(\xi)|^2 |\widehat{\phi}(2^{-j}\xi)|^2\, d\xi.\end{aligned}$$

Using the assumption that $\widehat{\phi}$ is continuous at 0, it is easy to show that $|\widehat{\phi}(2^{-j}\xi)| \to 1$ uniformly on $[-R, R]$ as $j \to \infty$. This implies that

$$\frac{1}{2\pi} \int_{-R}^{R} |\widehat{f}(\xi)|^2 |\widehat{\phi}(2^{-j}\xi)|^2\, d\xi \longrightarrow \frac{1}{2\pi} \int_{-R}^{R} |\widehat{f}(\xi)|^2\, d\xi = \int_{\mathbf{R}} |f(x)|^2\, dx$$

as $j \to \infty$. □

Example 8.5. With aid of Theorem 8.1, we will now show how the multiresolution analysis connected to the **Meyer wavelets** is constructed.

Suppose that the function θ is defined on $[0, 2\pi]$, is decreasing, and satisfies

$$0 \le \theta(\xi) \le 1 \quad \text{for } 0 \le \xi \le 2\pi, \tag{8.1}$$

$$\theta(\xi) = 1 \quad \text{for } 0 \le \xi \le \frac{2\pi}{3}, \tag{8.2}$$

$$\theta(\xi) + \theta(2\pi - \xi) = 1 \quad \text{for } 0 \le \xi \le 2\pi. \tag{8.3}$$

Notice that (8.2) and (8.3) imply that $\theta(\xi) = 0$ for $\frac{4\pi}{3} \le \xi \le 2\pi$. Then extend to $\mathbf{R}$ by first making an even extension on $[-2\pi, 2\pi]$ and then letting $\theta(\xi) = 0$ for $|\xi| > 2\pi$. Let the scaling function ϕ be defined by

$$\phi(x) = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \sqrt{\theta(\xi)} e^{ix\xi} \, d\xi, \quad x \in \mathbf{R},$$

so that $\widehat{\phi} = \sqrt{\theta}$.

Every $\xi \in \mathbf{R}$ has a unique representation modulo 2π as $\xi = 2m\pi + \xi_0$, where the quotient m is an integer and the residue ξ_0 satisfies $0 \le \xi_0 < 2\pi$. It follows that

$$\begin{aligned}\sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2 &= \sum_{k\in\mathbf{Z}} \theta(\xi_0 + 2(k+m)\pi) = \theta(\xi_0) + \theta(\xi_0 - 2\pi) \\ &= \theta(\xi_0) + \theta(2\pi - \xi_0) = 1\end{aligned}$$

because of the facts that θ is even and satisfies (8.3). According to Corollary 6.3, this shows that the sequence $(\phi_{0,k})_{k\in\mathbf{Z}}$ is orthonormal.

We next define the filter m_ϕ by first letting

$$m_\phi(\xi) = \sqrt{\theta(2\xi)}, \quad |\xi| \le \pi,$$

and then extending m_ϕ to $\mathbf{R}$ with period 2π. Then $m_\phi \in L^2(\mathbb{T})$. Since

$$\theta(\xi) = \theta(\xi)\theta(\tfrac{\xi}{2}) \quad \text{for every } \xi \in \mathbf{R},$$

the filter identity

$$\widehat{\phi}(\xi) = m_\phi(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2})$$

holds for every $\xi \in \mathbf{R}$. Thus, ϕ satisfies a scaling equation.

Finally, since $\theta(\xi) = 1$ for $0 \le \xi \le \frac{2\pi}{3}$, $\widehat{\phi}$ is continuous at 0 and $\widehat{\phi}(0) = 1$. Theorem 8.1 now shows that ϕ is the scaling function for a multiresolution analysis.

Notice that $\phi \in C^\infty(\mathbf{R})$ since $\widehat{\phi}$ has compact support. Also, if $\sqrt{\theta} \in C^\infty(\mathbf{R})$, then $\phi \in \mathscr{S}$.

Example 8.6. The Shannon system in Example 6.5 is in fact a special case of the more general Meyer systems. Indeed, if we choose θ so that $\theta(\xi) = 1$ for $|\xi| < \pi$, i.e., $\theta = \chi_{(-\pi,\pi)}$, then $\widehat{\phi} = \chi_{(-\pi,\pi)}$ and therefore

$$\phi(x) = \operatorname{sinc} \pi x, \quad x \in \mathbf{R}.$$

9. Construction of Wavelets

Definition 9.1. Given a a multiresolution analysis $(V_j)_{j\in\mathbf{Z}}$ of $L^2(\mathbf{R})$, let

$$W_j = V_{j+1} \ominus V_j$$

denote the orthogonal complement of V_j in V_{j+1} for $j \in \mathbf{Z}$. The spaces W_j are called **detail spaces**.

We leave the proof of the following proposition to the reader, which shows that the detail spaces W_j have the same scaling properties as the spaces V_j.

Proposition 9.2. *Suppose that $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$. Then, for every $j \in \mathbf{Z}$, $f(\,\cdot\,) \in W_j$ if and only if $f(2^{-j}\,\cdot\,) \in W_0$.*

Theorem 9.3. *Suppose that $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$. Then*

$$L^2(\mathbf{R}) = \bigoplus_{j\in\mathbf{Z}} W_j. \tag{9.1}$$

The right-hand side in (9.1) is the subspace of $L^2(\mathbf{R})$ that consists of all series

$$f = \sum_{j\in\mathbf{Z}} f_j$$

with $f_j \in W_j$ for $j \in \mathbf{Z}$, where the convergence is in $L^2(\mathbf{R})$.

Proof of Theorem 9.3. We will first show that

$$L^2(\mathbf{R}) = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j.$$

Denote the right-hand side in this identity by M. If a function $f \in L^2(\mathbf{R})$ belongs to $M^\perp$, then $f \perp V_0$ and $f \perp W_j$ for every integer $j \geq 0$. By the definition of W_j and the monotonicity of the multiresolution analysis, this implies that $f \perp V_j$ for every integer $j \in \mathbf{Z}$ and therefore that $f \perp \overline{\bigcup_{j\in\mathbf{Z}} V_j}$. Since $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$, it follows that $f = 0$.

We will finally show that

$$V_0 = \bigoplus_{j=1}^{\infty} W_{-j}.$$

Denote the right-hand side in this identity by N. If a function $f \in V_0$ belongs to $N^\perp$, then $f \perp W_{-j}$ for every integer $j \geq 1$. This then implies that $f \in V_{-j}$ for every integer $j \geq 0$ and therefore that $f \in \bigcap_{j=0}^{\infty} V_{-j}$. Since $\bigcap_{j=0}^{\infty} V_{-j} = \{0\}$, it follows that $f = 0$. □

Theorem 9.3 immediately gives the following important corollary.

Corollary 9.4. *Suppose that $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$. If there exists a function $\psi \in L^2(\mathbf{R})$ such that $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0, then $(\psi_{j,k})_{j,k\in\mathbf{Z}}$ is a wavelet basis for $L^2(\mathbf{R})$.*

Let $\phi \in V_0$ be the scaling function for a multiresolution analysis $(V_j)_{j\in\mathbf{Z}}$ of $L^2(\mathbf{R})$. For $\psi \in W_0$, let $m_\psi \in L^2(\mathbb{T})$ be the filter in Theorem 7.3, so that

$$\widehat{\psi}(\xi) = m_\psi(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2}) \quad \text{for a.e. every } \xi \in \mathbf{R}. \tag{9.2}$$

We then define the matrix M by

$$M(\xi) = \begin{pmatrix} m_\phi(\xi) & m_\psi(\xi) \\ m_\phi(\xi+\pi) & m_\psi(\xi+\pi) \end{pmatrix}, \quad \xi \in \mathbf{R}. \tag{9.3}$$

Theorem 9.5. *Suppose that $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$ with scaling function ϕ.*

(a) *If $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system W_0, then the matrix M in (9.3) is unitary a.e.*

(b) *Conversely, if M is unitary a.e. for some function $m_\psi \in L^2(\mathbb{T})$ and the function $\psi \in L^2(\mathbf{R})$ is defined by (9.2), then $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal sequence in W_0.*

Notice that the matrix M is unitary at $\xi \in \mathbf{R}$ if

$$|m_\phi(\xi)|^2 + |m_\phi(\xi+\pi)|^2 = 1, \tag{9.4}$$

$$|m_\psi(\xi)|^2 + |m_\psi(\xi+\pi)|^2 = 1, \tag{9.5}$$

$$m_\phi(\xi)\overline{m_\psi(\xi)} + m_\phi(\xi+\pi)\overline{m_\psi(\xi+\pi)} = 0. \tag{9.6}$$

Proof of Theorem 9.5. We first prove (a). So let us assume that $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in W_0 and deduce (9.4)–(9.6). Since $(\phi_{0,k})_{k\in\mathbf{Z}}$ is orthonormal, we know according to Corollary 6.3 that

$$\sum_{k\in\mathbf{Z}} |\widehat{\phi}(2\xi + 2k\pi)|^2 = 1$$

for a.e. $\xi \in \mathbf{R}$. We then apply the filter identity (7.4) for ϕ and split the series into sums over even and odd indices, using the fact that m_ϕ has period 2π:

$$\begin{aligned}\sum_{k\in\mathbf{Z}} |\widehat{\phi}(2\xi + 2k\pi)|^2 &= \sum_{l\in\mathbf{Z}} |m_\phi(\xi+2l\pi)|^2|\widehat{\phi}(\xi+2l\pi)|^2 \\ &\quad + \sum_{l\in\mathbf{Z}} |m_\phi(\xi+(2l+1)\pi)|^2|\widehat{\phi}(\xi+(2l+1)\pi)|^2 \\ &= |m_\phi(\xi)|^2 \sum_{l\in\mathbf{Z}} |\widehat{\phi}(\xi+2l\pi)|^2 \\ &\quad + |m_\phi(\xi+\pi)|^2 \sum_{l\in\mathbf{Z}} |\widehat{\phi}(\xi+\pi+2l\pi)|^2 \\ &= |m_\phi(\xi)|^2 + |m_\phi(\xi+\pi)|^2.\end{aligned}$$

This establishes (9.4). The proof of (9.5) is identical. Since $\psi_{0,k} \in W_0$, $\psi_{0,k}$ is orthogonal to every function $\phi_{0,l}$. Using this fact together with Plancherel's theorem, we obtain that

$$\int_{-\infty}^{\infty} \psi(x-k)\overline{\phi(x-l)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}e^{-i(k-l)\xi}\,d\xi = 0$$

for all $k, l \in \mathbf{Z}$. Put $n = k - l$ and make a periodization of the last integral:

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}e^{-in\xi}\,d\xi &= \sum_{k\in\mathbf{Z}} \int_{2k\pi}^{2(k+1)\pi} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}e^{-in\xi}\,d\xi \\ &= \sum_{k\in\mathbf{Z}} \int_0^{2\pi} \widehat{\psi}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}e^{-in\xi}\,d\xi \\ &= \int_0^{2\pi} \Big(\sum_{k\in\mathbf{Z}} \widehat{\psi}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}\Big)e^{-in\xi}\,d\xi.\end{aligned}$$

This shows that all Fourier coefficients of the 2π-periodic function inside the brackets are all zero, and hence that

$$\sum_{k\in\mathbf{Z}} \widehat{\psi}(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)} = 0$$

for a.e. $\xi \in \mathbf{R}$. We then replace ξ by 2ξ in the last identity and continue as in the first part of the proof:

$$\begin{aligned}\sum_{k\in\mathbf{Z}} \widehat{\psi}(2\xi+2k\pi)\overline{\widehat{\phi}(2\xi+2k\pi)} &= \sum_{l\in\mathbf{Z}} m_\phi(\xi+2l\pi)\overline{m_\psi(\xi+2l\pi)}|\widehat{\phi}(\xi+2l\pi)|^2\\ &+\sum_{l\in\mathbf{Z}} m_\phi(\xi+(2l+1)\pi)\overline{m_\psi(\xi+(2l+1)\pi)}|\widehat{\phi}(\xi+(2l+1)\pi)|^2\\ &= m_\phi(\xi)\overline{m_\psi(\xi)} + m_\phi(\xi+\pi)\overline{m_\psi(\xi+\pi)} = 0\end{aligned}$$

for a.e. $\xi \in \mathbf{R}$, which proves (9.6).

We now turn our attention to the proof of (b). Notice that all steps in the calculations above can be reversed. This means that if (9.4)–(9.6) hold, then $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in W_0. □

We will now show how the function ψ can be constructed with the aid of Theorem 9.5. The orthogonality condition (9.6) gives that

$$\begin{pmatrix} m_\psi(\xi) \\ m_\psi(\xi+\pi)\end{pmatrix} = \alpha(\xi)\begin{pmatrix} \overline{m_\phi(\xi+\pi)} \\ -\overline{m_\phi(\xi)}\end{pmatrix} \tag{9.7}$$

for some function α on $\mathbf{R}$. Since the vectors in the left- and right-hand side both have length 1, we see that

$$|\alpha(\xi)| = 1 \quad \text{for a.e. } \xi \in \mathbf{R}. \tag{9.8}$$

Moreover, replacing ξ by $\xi+\pi$ in (9.7), using the fact that both m_ϕ and m_ψ have period 2π, we have that

$$\begin{pmatrix} m_\psi(\xi+\pi) \\ m_\psi(\xi)\end{pmatrix} = \alpha(\xi+\pi)\begin{pmatrix} \overline{m_\phi(\xi)} \\ -\overline{m_\phi(\xi+\pi)}\end{pmatrix}.$$

This identity in combination with (9.7) shows that

$$\alpha(\xi+\pi) = -\alpha(\pi) \quad \text{for } a.e.\ \xi \in \mathbf{R}. \tag{9.9}$$

One function, that satisfies (9.8) and (9.9), is

$$\alpha(\xi) = -e^{-i\xi}, \quad \xi \in \mathbf{R}.$$

With this choice, we obtain that

$$\begin{aligned} m_\psi(\xi) = \alpha(\xi)\overline{m_\phi(\xi+\pi)} &= -\tfrac{\sqrt{2}}{2}e^{-i\xi}\overline{\sum_{k\in\mathbf{Z}} c_k e^{-ik(\xi+\pi)}}\\ &= -\tfrac{\sqrt{2}}{2}\sum_{k\in\mathbf{Z}}(-1)^k\overline{c_k}e^{-i(1-k)\xi}\\ &= \tfrac{\sqrt{2}}{2}\sum_{l\in\mathbf{Z}}(-1)^l\overline{c_{1-l}}e^{-il\xi},\end{aligned}$$

and therefore that

$$\psi(x) = \sum_{k\in\mathbf{Z}}(-1)^k\overline{c_{1-k}}\phi_{1,k}(x), \quad x \in \mathbf{R}.$$

It remains to be shown that ψ generates an orthonormal basis for W_0.

Theorem 9.6. *Suppose that $(V_j)_{j\in\mathbf{Z}}$ is a multiresolution analysis of $L^2(\mathbf{R})$ with scaling function ϕ. If $\psi \in L^2(\mathbf{R})$ is defined by*

$$\psi(x) = \sum_{k\in\mathbf{Z}} (-1)^k \overline{c_{1-k}} \phi_{1,k}(x), \tag{9.10}$$

where $(c_k)_{k\in\mathbf{Z}}$ are the structure constants in (7.3)*, or equivalently by*

$$\widehat{\psi}(\xi) = e^{-i(\xi/2+\pi)} \overline{m_\phi(\tfrac{\xi}{2}+\pi)} \widehat{\phi}(\tfrac{\xi}{2}),$$

then $(\psi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal basis for W_0.

Proof. Suppose that f is a function in W_0 with filter identity

$$\widehat{f}(\xi) = m_f(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2});$$

see Theorem 7.3. As in the discussion above, the fact that f is orthogonal to V_0 means that

$$m_f(\xi) = \beta(\xi)\overline{m_\phi(\xi+\pi)},$$

where $\beta \in L^2(\mathbb{T})$ satisfies $\beta(\xi+\pi) = -\beta(\xi)$ for a.e. $\xi \in \mathbf{R}$. Recall that

$$\widehat{\psi}(\xi) = e^{-i(\xi/2+\pi)} \overline{m_\phi(\tfrac{\xi}{2}+\pi)} \widehat{\phi}(\tfrac{\xi}{2}).$$

It therefore follows that

$$\widehat{f}(\xi) = e^{i(\xi/2+\pi)}\beta(\tfrac{\xi}{2})\widehat{\psi}(\xi).$$

Let us define

$$\alpha(\xi) = e^{i(\xi/2+\pi)}\beta(\tfrac{\xi}{2}), \quad \xi \in \mathbf{R}.$$

Then α has period 2π since

$$\alpha(\xi+2\pi) = e^{i(\xi/2+2\pi)}\beta(\tfrac{\xi}{2}+\pi) = -e^{i\xi/2}\beta(\tfrac{\xi}{2}) = e^{i(\xi/2+\pi)}\beta(\tfrac{\xi}{2}) = \alpha(\xi)$$

for a.e. $\xi \in \mathbf{R}$. Also,

$$\begin{aligned}\int_0^{2\pi} |\alpha(\xi)|^2\,d\xi &= \int_0^{2\pi} |\beta(\tfrac{\xi}{2})|^2\,d\xi \\ &= 2\int_0^{\pi} |\beta(\xi)|^2(|m_\phi(\xi)|^2 + |m_\phi(\xi+\phi)|^2)\,d\xi \\ &= 2\int_0^{2\pi} |\beta(\xi)|^2|m_\phi(\xi+\phi)|^2\,d\xi \\ &= 2\int_0^{2\pi} |m_f(\xi)|^2\,d\xi < \infty.\end{aligned}$$

This shows that $\alpha \in L^2(\mathbb{T})$. Thus,

$$\widehat{f}(\xi) = \alpha(\xi)\widehat{\psi}(\xi)$$

or equivalently that

$$f(x) = \sum_{k\in\mathbf{Z}} a_k\psi(x-k) = \sum_{k\in\mathbf{Z}} a_k\psi_{0,k}(x)$$

in $L^2(\mathbf{R})$. Hence, $(\psi_{0,k})_{k\in\mathbf{Z}}$ is a basis for W_0. □

Example 9.7.

10. Regular Wavelets with Compact Support

Suppose that ϕ is the scaling function for a multiresolution analysis and moreover that $\operatorname{supp}\phi \subset [-R, R]$ for some $R > 0$ and $\widehat{\phi}(0) \neq 0$. As we have seen, ϕ then satisfies the scaling equation

$$\phi(x) = \sum_{k\in\mathbf{Z}} c_k \phi_{1,k}(x) \quad \text{in } L^2(\mathbf{R}),$$

which in turn is equivalent to the identity

$$\widehat{\phi}(\xi) = m(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2}) \quad \text{a.e.},$$

where the filter m is given by

$$m(\xi) = \tfrac{\sqrt{2}}{2} \sum_{k\in\mathbf{Z}} c_k e^{-ik\xi}, \quad \xi \in \mathbf{R}.$$

Notice that

$$c_k = \sqrt{2}\int_{-R}^{R} \phi(x)\overline{\phi(2x-k)}\,dx = 0 \quad \text{for } |k| \geq 3R,$$

which shows that m is a trigonometric polynomial. A direct consequence of this observation is the fact that the wavelet ψ, given by

$$\psi(x) = \sum_{k\in\mathbf{Z}} (-1)^k \overline{c_{1-k}} \phi_{1,k}(x), \quad x \in \mathbf{R},$$

has compact support.

Iterating in the filter identity, we see that

$$\widehat{\phi}(\xi) = m(\tfrac{\xi}{2})\widehat{\phi}(\tfrac{\xi}{2}) = m(\tfrac{\xi}{2})m(\tfrac{\xi}{4})\widehat{\phi}(\tfrac{\xi}{4}) = \dots = \widehat{\phi}(\tfrac{\xi}{2^n}) \prod_{j=1}^{n} m(\tfrac{\xi}{2^j}) \quad \text{a.e.}$$

for $n = 1, 2, \dots$. Since ϕ has compact support, ϕ belongs to $L^1(\mathbf{R})$ and therefore is $\widehat{\phi}$ continuous on $\mathbf{R}$. Using this together with the assumption that $\widehat{\phi}(0) \neq 0$, we see that we can let n tend to infinity in the last identity and obtain that

$$\widehat{\phi}(\xi) = \widehat{\phi}(0) \prod_{j=1}^{\infty} m(\tfrac{\xi}{2^j}) \quad \text{a.e.}$$

We summarize these observations in the following theorem.

Theorem 10.1. *Suppose that the ϕ scaling function for a multiresolution analysis has compact support and $\widehat{\phi}(0) \neq 0$. Then the filter m is a trigonometric polynomial, the corresponding wavelet ψ has compact support, and*

$$\widehat{\phi}(\xi) = \widehat{\phi}(0) \prod_{j=1}^{\infty} m(\tfrac{\xi}{2^j}) \quad \text{a.e.}$$

We will now show that this argument can essentially be reversed. To this end, let m be a trigonometric polynomial of the form

$$m(\xi) = \frac{\sqrt{2}}{2} \sum_{k=-l}^{l} c_k e^{-ik\xi}, \quad \xi \in \mathbf{R} \tag{10.1}$$

such that

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1 \quad \text{for every } \xi \in \mathbf{R}, \tag{10.2}$$

$$m(0) = 1, \tag{10.3}$$

$$m(\xi) \neq 0 \quad \text{for } \xi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]. \tag{10.4}$$

Proposition 10.2. *Suppose that m is a trigonometric polynomial of the form* (10.1) *that satisfies conditions* (10.2)–(10.4)*. Then the product*

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m(\tfrac{\xi}{2^j}), \quad \xi \in \mathbf{R}, \tag{10.5}$$

converges locally uniformly on $\mathbf{R}$ and satisfies $\widehat{\phi}(0) = 1$. Moreover, $\widehat{\phi}$ is a continuous function.

Proof. Let P_n denote the n-th partial product of the product (10.5), i.e.,

$$P_n(\xi) = \prod_{j=1}^{n} m(\tfrac{\xi}{2^j}), \quad \xi \in \mathbf{R},\ n = 1, 2, \dots .$$

Then

$$|m(\xi) - 1| = |m(\xi) - m(0)| \leq \frac{\sqrt{2}}{2} \sum_{k=-l}^{l} |c_k||e^{-ik\xi} - 1| \leq \Big(\frac{\sqrt{2}}{2} \sum_{k=-l}^{l} |k||c_k| \Big) |\xi|$$

for every $\xi \in \mathbf{R}$. Using this observation, we obtain that

$$|P_{k+1}(\xi) - P_k(\xi)| = |m(\tfrac{\xi}{2^{k+1}}) - 1||P_k(\xi)| \leq C 2^{-(k+1)} |\xi|.$$

It follows that if $m > n$, then

$$|P_m(\xi) - P_n(\xi)| \leq C|\xi| \Big(\frac{1}{2^m} + \dots + \frac{1}{2^{n+1}} \Big) \leq C 2^{-n} |\xi|.$$

This shows that the product (10.5) converges uniformly on every compact subset of $\mathbf{R}$ and therefore represents a continuous function. Finally, $\widehat{\phi}(0) = 1$ according to (10.3). □

We will omit the proof of the following theorem. A consequence of this theorem is the fact that the function ϕ, implicitly defined by (10.5), belongs to $L^2(\mathbf{R})$.

Theorem 10.3. *Suppose that m is a trigonometric polynomial that satisfies conditions* (10.2)–(10.4)*. Then the function $\widehat{\phi}$ in* (10.5) *belongs to $L^2(\mathbf{R})$.*

The next step is to show that $\widehat{\phi}$ satisfies a filter equation. This means that ϕ also satisfies a scaling equation.

Proposition 10.4. *Suppose that m is a trigonometric polynomial that satisfies conditions* (10.2)–(10.4)*. Then*

$$\widehat{\phi}(\xi) = m(\tfrac{\xi}{2}) \widehat{\phi}(\tfrac{\xi}{2}) \quad a.e.$$

Proof. By the definition of $\widehat{\phi}$,

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m(\tfrac{\xi}{2^j}) = m(\tfrac{\xi}{2}) \prod_{j=2}^{\infty} m(\tfrac{\xi}{2^j}) = m(\tfrac{\xi}{2}) \widehat{\phi}(\tfrac{\xi}{2}) \quad \text{a.e.}$$ □

The construction also implies that ϕ has compact support. A short proof can be obtained by using some distributiton theory.

Theorem 10.5. *Suppose that m is a trigonometric polynomial of the form* (10.1) *that satisfies conditions* (10.2)–(10.4)*. Then* $\operatorname{supp}\phi \subset [-l, l]$.

Proof. Put

$$u_j = \frac{\sqrt{2}}{2} \sum_{k=-l}^{l} c_k \delta_{2^{-j}k}, \quad j = 1, 2, \dots,$$

where $\delta_{2^{-j}k}$ is the Dirac delta at $2^{-j}k$, considered as an element of $\mathscr{S}'$. It is not so hard to verify that

$$\widehat{u_j}(\xi) = m(\tfrac{\xi}{2^j}) \quad \text{for } \xi \in \mathbf{R}.$$

Also,

$$u_1 * \dots * u_n = \Big(\frac{\sqrt{2}}{2}\Big)^n \sum_{|k_j| \le l} c_{k_1} \cdot \dots \cdot c_{k_n} \delta_{2^{-1}k_1 + \dots + 2^{-n}k_n}.$$

Notice that the support of the convolution is a subset of the interval $[-l, l]$. Moreover,

$$\widehat{u_1 * \dots * u_n} = \prod_{j=1}^{n} m(\tfrac{\xi}{2^j}) \quad \text{for } n = 1, 2, \dots .$$

We know that the right-hand side of this identity converges to $\widehat{\phi}(\xi)$ locally uniformly. This together with the fact that the absolut value of each partial product is uniformly bounded by 1 implies that the convergence actually holds in $\mathscr{S}'$. It follows that $u_1 * \dots * u_n \to \phi$ in $\mathscr{S}'$ and therefore that $\operatorname{supp}\phi \subset [-l, l]$. □

We will finally show that $(\phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in $L^2(\mathbf{R})$ and begin with a lemma.

Lemma 10.6. *Suppose that m is a trigonometric polynomial of the form* (10.1) *that satisfies conditions* (10.2)–(10.4)*. Then the function*

$$\Phi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2, \quad \xi \in \mathbf{R},$$

is a trigonometric polynomial.

Proof. Plancherel's formula and a periodization argument shows that

$$\int_{\mathbf{R}} \phi(x)\overline{\phi(x-n)}\, dx = \frac{1}{2\pi} \int_{\mathbf{R}} |\widehat{\phi}(\xi)|^2 e^{in\xi}\, d\xi = \frac{1}{2\pi} \int_0^{2\pi} \Phi(\xi) e^{in\xi}\, d\xi.$$

The left-hand side is nonzero for not more than a finite number of integers n, which implies that all but a finite number of Fourier coefficients of Φ are 0 and therefore that Φ is a trigonometric polynomial. □

Theorem 10.7. *Suppose that m is a trigonometric polynomial of the form* (10.1) *that satisfies conditions* (10.2)–(10.4)*. Then $(\phi_{0,k})_{k\in\mathbf{Z}}$ is an orthonormal system in $L^2(\mathbf{R})$.*

Proof. The same argument that was used in the proof of Theorem 9.5 shows that

$$\Phi(2\xi) = |m(\xi)|^2\Phi(\xi) + |m(\xi+\pi)|^2\Phi(\xi+\pi) \quad \text{for every } \xi \in \mathbf{R}.$$

Let $a = \Phi(\xi_0)$ be the minimal value of Φ on $[-\pi, \pi]$. Then

$$a = \Phi(\xi_0) = |m(\tfrac{\xi_0}{2})|^2\Phi(\tfrac{\xi_0}{2}) + |m(\tfrac{\xi_0}{2}+\pi)|^2\Phi(\tfrac{\xi_0}{2}+\pi) \geq a.$$

Due to (10.2) and (10.4), this shows that $\Phi(\frac{\xi_0}{2}) = a$. Iterating this argument, we obtain that $\Phi(\frac{\xi_0}{2^j}) = a$ for $j = 0, 1, ...$ and therefore that $\Phi(0) = a$. Any nonzero integer k can be written in the form $k = 2^r s$, where r is a nonnegative integer and s is an odd integer. Using this together with the fact that $m(s\pi) = 0$, we obtain that

$$\widehat{\phi}(2k\pi) = \prod_{j=1}^{\infty} m(2^{r+1-j}s\pi) = 0,$$

and therefore that

$$\Phi(0) = \sum_{k\in\mathbf{Z}} |\widehat{\phi}(2k\pi)|^2 = |\widehat{\phi}(0)|^2 = 1.$$

This shows that $a = 1$.

A similar argument shows that the maximal value of Φ on $[-\pi, \pi]$ is 1 and hence that Φ is identically 1 on $[-\pi, \pi]$. Due to periodicity of Φ, this means that Φ is identically 1 on $\mathbf{R}$. □

Corollary 10.8. *Suppose that m is a trigonometric polynomial of the form* (10.1) *that satisfies conditions* (10.2)–(10.4). *Then ϕ generates a multiresolution analysis of $L^2(\mathbf{R})$ such that both ϕ and ψ have compact support.*

Our final result shows that it is possible to construct compactly supported wavelets that are as smooth as we want.

Theorem 10.9. *For every nonnegative integer r, there exists a multiresolution analysis of $L^2(\mathbf{R})$ such that both the scaling function ϕ and the wavelet ψ belong to $C_c^r(\mathbf{R})$.*

Sketch of proof. Choose a trigonometric polynomial m satisfies (10.2)–(10.4) and

$$\prod_{j=1}^{\infty} |m(\tfrac{\xi}{2^j})| \leq \frac{C}{(1+|\xi|)^{r+2}} \quad \text{for every } \xi \in \mathbf{R}.$$

Then the scaling function ϕ, given by

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \prod_{j=1}^{\infty} m(\tfrac{\xi}{2^j}) e^{i\xi x}\, d\xi, \quad x \in \mathbf{R},$$

belongs to $C_c^r(\mathbf{R})$. It follows from the formula (9.10) that the wavelet ψ also belongs to $C_c^r(\mathbf{R})$. □

Remark 10.10. One can show that there does not exist a multiresolution analysis of $L^2(\mathbf{R})$ such that both ϕ and ψ belong to $C_c^\infty(\mathbf{R})$.

Department of Mathematics, Linköping University, SE-581 83 Sweden