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ODEs and Dynamical Systems (TATA71)
Course programme, fall 2024

General information

Ordinary Differential Equations and Dynamical Systems (TATA71) is an optional course for MAT2, Y4, M4, EMM4. It is given in the second half of the fall semester (period ht2). All information is publicly available on the course webpage courses.mai.liu.se/GU/TATA71/. Lisam is currently not used in this course.

Literature

D. K. Arrowsmith & C. M. Place, *Dynamical Systems: Differential Equations, Maps, and Chaotic Behaviour*, Chapman and Hall/CRC (1992), ISBN 9780412390807. Available as an e-book via LiU's library.

Prerequisites

The prerequisites are basic courses in **single-variable** and **multi-variable calculus**, plus **linear algebra**. An honours course in **real analysis** may be helpful for understanding certain subtle details, but that's optional.

Here "single-variable calculus" means that you are supposed to already have seen the basic techniques for solving simple ODEs (as taught, for example, in the course TATA42 here at LiU): separation of variables, integrating factors, the characteristic polynomial, the method of undetermined coefficients, and so on. However, we will spend some time brushing up on this at the beginning of the course.

Teaching

As you can see in the table of contents on the first page, the material is organized into 10 lectures. The first classroom session, "Lecture 1" at the very beginning of the study period, is an ordinary lecture. The rest of the course follows a "flipped classroom" format, where Lectures 2–10 are pre-recorded video lectures that you are supposed to watch in advance, before the corresponding classroom session, which is marked "Seminar" in the online schedule, TimeEdit. In class, there will then be a short summary instead of a regular lecture, and the rest of the time will be available for discussing the theory, looking at additional examples, working on the exercises, and so on. There are also four "Lessons", which provide some extra time for catching up, plus an optional video lecture with some "outlook" material for those who are interested. Attendance is **not** mandatory (although it is recommended, of course).

Course evaluations and changes

Previous course evaluations can be found by searching Evaliuate for "TATA71". Compared to last year, the contents of the course are unchanged, but there have been some updates to the course programme and to the selection of exercises, including the homework problems. The most noticable change is perhaps that there are now full solutions available for some of the exercises, which is something that previous students have asked for.

[Table of contents]

Examination

The examination consists of two parts:

- **UPG1. Homework assignments, worth 2 hp (= 2 ECTS credits).**

  Some of the exercises are assigned as homework problems (marked with yellow in this course programme), to be handed in continually during the course. These problems are only graded pass/fail, and if you fail a problem, you simply hand in a corrected version later. Discussing the problems with the teacher and with your fellow students is allowed, but please write the solution in your own words; it's not allowed to just copy someone else's solution!

  The solutions should be handed in **on paper**, either directly to me in class or at my office at the math department in building B (room 3A:666), or else in my pigeon-hole messagebox at the northern end of the same corridor. Handwritten solutions are fine, and you can write them in English or in Swedish. But please **do not write in red**, since I'll be using a red pen when marking. And only write on **one side** of the paper.

  The deadline for completing the homework assignments is **Jan 15, 2025**, which is the day before the first written exam. I strongly recommend that you hand in the last problems **before the Christmas break**, to make sure that there is enough time for getting feedback (and making corrections, if necessary) before the deadline.

- **TEN1. A written test (5 hours), worth 4 hp.**

  The test contains 6 problems, each of which is graded as pass (3 or 2 points) or fail (1 or 0 points). The total grade for the course is determined by the grade for the written test, which in turn is determined as follows: for grade 3/4/5 (respectively), you need 3/4/5 passed problems and in addition at least 8/11/14 points in total.

  The written examination takes place on LiU's Campus Valla in Linköping, three times per year (January, March, August). Under normal circumstances, exams at other locations will *not* be arranged.

[Table of contents]

What's this course about?

As the name of the course suggests, we will study **ODEs** and **dynamical systems**.

- **ODE** is a standard abbreviation for **ordinary differential equation**, where the function that we seek depends on *one* variable. So an ODE is just a good old differential equation like those which you have already seen in your single-variable calculus course. We will also encounter *systems* of ODEs, involving several unknown functions at once, but each function will still only depend on one variable.

  In contrast, a **partial differential equation** (**PDE**) is a differential equation where one seeks a function depending on *several* variables, so that *partial derivatives* come into play. But that's a different subject; see the course TATA27.

- The idea of a **dynamical system** is rather broad, and it is hard to give a precise mathematical definition which would cover every possible use of the phrase. But it refers to a "system" (whatever that is) which changes in a deterministic way as time passes, and which is "memoryless" in the sense that the future of the system, at any given instant, is uniquely determined by the present state alone; the past is irrelevant.

  We will usually assume when talking about dynamical systems that the laws governing the evolution don't change with time. That is, if we start the system in a given state T units of time from now, we will get the same evolution as if we start it in that state right now (except that everything will be delayed by T time units of course). If this needs to be emphasized, one uses the phrase **autonomous dynamical system** – a system which "runs on its own", in contrast to **non-autonomous** dynamical systems where there may be some external time-dependent factors which influence the evolution of the system.

  The state of the system is represented mathematically by an element of a set called the **state space** or the **phase space**, typically $\mathbf{R}^n$, or maybe some subset of $\mathbf{R}^n$ like a cylinder, a sphere, or a torus. So one pictures the evolution of the system as the motion of a point in the state space.

- A **discrete-time dynamical system** is one where things happen at distinct time steps. We can use integers to label the time steps, so that the system is in the state $x_n \in S$ at time $n \in \mathbf{Z}$, where S is the state space. Then the evolution of the system is simply specified by some function $f\colon S \to S$, like this:

  $$x_{n+1} = f(x_n), \qquad n \in \mathbf{Z}.$$

  (The system is autonomous since the function f is the same for all n.)

  Discrete-time dynamical systems are a very important part of the general theory of dynamical systems, but will not be encountered very much in this course.

- In a **continuous-time dynamical system**, time passes smoothly, so we use real numbers to describe time, and talk about the system being in the state $x(t)$ at time $t \in \mathbf{R}$. In this case, if the state space is $\mathbf{R}^n$ for simplicity, the evolution is determined by a system of first-order ODEs for the state $\mathbf{x}(t) = \big(x_1(t), \dots, x_n(t)\big)$:

  $$\begin{aligned} dx_1/dt &= f_1(x_1, \dots, x_n), \\ dx_2/dt &= f_2(x_1, \dots, x_n), \\ &\ \ \vdots \\ dx_n/dt &= f_n(x_1, \dots, x_n), \end{aligned}$$

  or simply $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x})$ for short, with $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{f}\colon \mathbf{R}^n \to \mathbf{R}^n$. (The system is autonomous since the function $\mathbf{f}$ doesn't depend on t.)

  In order for this to really define a dynamical system, we must impose suitable conditions on $\mathbf{f}$ which will guarantee **existence** and **uniqueness** of the solution to the ODEs for a given initial condition, so that the future is uniquely determined by the present state.

[Table of contents]

Lecture 1. Basics of first-order ODEs

(Arrowsmith & Place, sections 1.1, 1.2. And your old calculus textbook, if needed.)

Since this first meeting takes place at the very beginning of the study period, I will not expect to you have watched the video lecture in advance. So it will be more like a regular lecture, in the classroom. Some basic ideas are introduced:

- **Existence and uniqueness theorems** for first-order ODEs $\frac{dx}{dt} = X(t,x)$.

  Proposition 1.1.1 is usually called **Peano's existence theorem**.

  Proposition 1.1.2 is a slightly simplified version of the **Picard–Lindelöf theorem**. (The assumption "$\partial X/\partial x$ exists and is continuous" is stronger than necessary; we'll study this more thoroughly in Lecture 9.)

- How to find **explicit solutions** in simple cases.

  (Hopefully, this will mostly be a question of remembering methods that you have learned in previous courses. See the summary below.)

- How to sketch the solution's **graph** $x = x(t)$ in the xt-plane directly from a first-order ODE $\dot{x} = X(x,t)$.

- How to draw the **phase portrait** on the x-axis for a one-dimensional (continuous-time & autonomous) dynamical system $\dot{x} = X(x)$.

A comment about notation

You might be used to ODEs looking something like this:

$$y''(x) + 5y'(x) + 4y(x) = \cos x \qquad \text{(or simply} \quad y'' + 5y' + 4y = \cos x),$$

where the independent variable is called x, and $y = y(x)$ is the function that we seek. But since this is a course about ODEs with a "dynamical systems perspective", we will instead call the independent variable t, for "time", and use names like $x(t)$ or $y(t)$ for the sought functions. So the same ODE now instead looks as follows:

$$x''(t) + 5x'(t) + 4x(t) = \cos t \qquad \text{or} \qquad \ddot{x}(t) + 5\dot{x}(t) + 4x(t) = \cos t.$$

(It is common to use **dots** instead of primes to denote **derivatives with respect to time**.)

Two very fundamental examples

- **Exponential growth/decay**:

  $$x'(t) = r\,x(t).$$

  This is a linear equation, and we can solve it in several ways: integrating factor, characteristic polynomial, separation of variables. Either way, the solution with initial condition $x(0) = x_0$ is

  $$x(t) = x_0\,e^{rt}.$$

  This will be encountered again and again in this course, and you will be expected to *instantly* recognize this equation and know its solution. Phase portrait (if $r > 0$): "$\longleftarrow 0 \longrightarrow$".

- The **logistic equation** with growth rate r and carrying capacity K:

  $$x'(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right).$$

This nonlinear equation is often solved via separation of variables (followed by integration using partial fractions), but an easier way is to use the substitution $x(t) = 1/y(t)$, since this is a Bernoulli equation (see below). Solution, with $x(0) = x_0$:

$$x(t) = \frac{K x_0}{x_0 + (K - x_0)\, e^{-rt}} = \frac{K x_0\, e^{rt}}{K + (e^{rt} - 1)x_0}.$$

Here you don't need to memorize the solution formula, but you should be able to derive it, and you should also know roughly what the *graph* of the solution $x(t)$ looks like for different values of x_0. (In particular, $x(t) = 0$ and $x(t) = K$ are constant solutions.) Phase portrait (if $r > 0$ and $K > 0$): "$\longleftarrow 0 \longrightarrow K \longleftarrow$".

Summary of some exact solution methods

- **Linear first order equations** $x' + ax = b$, where the coefficients a and b may be functions of the time variable t:

  $$x'(t) + a(t)\, x(t) = b(t).$$

  How to solve: Find an antiderivative of $a(t)$; call it $A(t)$. Then multiply both sides of the ODE by the **integrating factor** $e^{A(t)}$, and use the product rule for derivatives (backwards). This gives

  $$\left(e^{A(t)}\, x(t)\right)' = e^{A(t)}\, b(t),$$

  which can now be integrated.

- **Separable equations** $f(x)\, x' = g(t)$.

  What this means is that we seek $x(t)$ such that

  $$f(x(t))\, x'(t) = g(t).$$

  Integrating this with respect to t, using the chain rule (backwards), we immediately obtain the solution in the implicit form

  $$F(x(t)) = G(t) + C,$$

  where $F(x)$ is some antiderivative of $f(x)$ and $G(t)$ is some antiderivative of $g(t)$. Usually this is remembered via the trick of writing $x' = dx/dt$ and "separating the variables" by "multiplying by dt", and then attaching integral signs:

  $$f(x)\,\frac{dx}{dt} = g(t) \qquad \Longleftrightarrow \qquad \int f(x)\, dx = \int g(t)\, dt.$$

  It's sometimes convenient to use *definite* integrals instead, particularly if we want to find a solution satisfying a given **initial condition** $x(t_0) = x_0$:

  $$\int_{x_0}^{x(t)} f(\xi)\, d\xi = \int_{t_0}^{t} g(\tau)\, d\tau,$$

  or in other words

  $$F(x(t)) - F(x_0) = G(t) - G(t_0).$$

  As an important special case of separable equations we have **one-dimensional dynamical systems** $x' = X(x)$, which can always be solved (in principle) by separation of variables:

  $$x' = X(x) \quad \Longleftrightarrow \quad x(t) = x^* \quad \text{where } X(x^*) = 0$$
  $$\text{or} \quad \int \frac{dx}{X(x)} = \int dt = t + C.$$

  Warning! This method is full of pitfalls! When dividing by $X(x)$, don't forget to consider the case $X(x) = 0$ separately; there is a constant solution $x(t) = x^*$ for each zero x^* of the function $X(x)$.

And one also needs to be very careful with handling logarithms and absolute values correctly when integrating, and when simplifying the solution. So if the ODE can be solved by some other method, it may be wise to try that method first.

(In addition to this, there may also be subtle problems having to do with non-uniqueness of solutions, if $X(x)$ isn't nice enough. One typical such example, which is mentioned in the lecture, is $\dot{x} = 2\sqrt{|x|}$ with $x(0) = 0$.)

- **Linear equations of arbitrary order.**

  A linear ODE of order n has the form

  $$x^{(n)}(t) + a_{n-1}(t)\,x^{(n-1)}(t) + \cdots + a_2(t)\,x''(t) + a_1(t)\,x'(t) + a_0(t)\,x(t) = b(t).$$

  The general solution of such an equation has the structure

  $$x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t)$$

  where $x_{\text{part}}(t)$ is a **particular solution** and $x_{\text{hom}}(t)$ (called the **homogeneous solution** or the **complementary solution**) is the general solution of the corresponding homogeneous equation which has 0 instead of $b(t)$ on the right-hand side.

  Solving higher-order equations with **time-dependent coefficients** is usually a rather hopeless task, except that one may try to find solutions in the form of a **power series** in t.

- **Linear equations of arbitrary order with constant coefficients.**

  The problem becomes tractable when the coefficients $a_k(t) = c_k$ are time-independent:

  $$x^{(n)}(t) + c_{n-1}x^{(n-1)}(t) + \cdots + c_2 x''(t) + c_1 x'(t) + c_0 x(t) = b(t).$$

  Then $x_{\text{hom}}(t)$ can be found by looking at the roots of the **characteristic polynomial**

  $$p(r) = r^n + c_{n-1}\,r^{n-1} + \cdots + c_2 r^2 + c_1 r + c_0.$$

  If the roots are real and simple, it's straightforward to write down $x_{\text{hom}}(t)$. For repeated and/or complex roots, the rules for constructing $x_{\text{hom}}(t)$ are a bit more complicated.

  To find $x_{\text{part}}(t)$ one usually employs "the method of undetermined coefficients", which consists in making a suitable *ansatz*, i.e., predicting what form the solution will take, based on what the right-hand side looks like, and then adjusting the values of the free parameters in the ansatz in order to actually satisfy the ODE. For not-too-complicated right-hand sides (polynomials, exponentials, sine/cosine functions) there are rules for how to make an ansatz which is guaranteed to work. For other right-hand sides, one would have to make a guess and hope for the best.[1]

  See your calculus textbook (or some other source) for details about all this.

- A **Bernoulli equation** is an ODE of the form

  $$x'(t) + p(t)\,x(t) = q(t)\,x(t)^k$$

  where k is a constant (not necessarily an integer). Note that this is a **nonlinear** ODE (unless $k = 0$ or $k = 1$) because of the expression $x(t)^k$ on the right-hand side. The nonzero solutions to such an equation can be found by dividing both sides by that factor $x(t)^k$, to get

  $$\frac{x'(t) + p(t)\,x(t)}{x(t)^k} = q(t),$$

  or in other words

  $$x'(t)\,x(t)^{-k} + p(t)\,x(t)^{1-k} = q(t).$$

[1] In Lecture 10 we will learn a more powerful tool, "the method of variation of constants", which in principle can handle any right-hand side, without any guessing.

Indeed, just let

$$y(t) = x(t)^{1-k},$$

which according to the chain rule has the derivative

$$y'(t) = (1-k)x(t)^{-k}x'(t),$$

and compare this to what we had in our equation:

$$\underbrace{x'(t)\,x(t)^{-k}}_{=y'(t)/(1-k)} + p(t)\,\underbrace{x(t)^{1-k}}_{=y(t)} = q(t).$$

That is, in terms of the new unknown function $y(t)$ we get a first-order **linear** ODE, the type that can be solved using an integrating factor:

$$\frac{1}{1-k}\,y'(t) + p(t)\,y(t) = q(t).$$

- Later in this course we will thoroughly study **linear first order systems**

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t),$$

where $\mathbf{x}(t) = \big(x_1(t), \dots, x_n(t)\big)^T$ is a column vector of functions and $A(t)$ is matrix of size $n \times n$. But you may already have seen in your linear algebra course how to solve such a system in the case where A is a **constant** matrix which happens to be **diagonalizable**. In this case, the change of variables $\mathbf{x} = M\mathbf{y}$, where the columns of M are n linearly independent eigenvectors of A, leads to a decoupled system $\mathbf{y}'(t) = D\mathbf{y}(t)$, where D is a diagonal matrix with the eigenvalues of A along the diagonal (in the same order as the corresponding eigenvectors were listed in the matrix M).

- **Computer algebra systems** can sometimes be of use for finding exact solutions to ODEs. Here are some examples of the syntax used in Mathematica (see DSolve documentation):

  DSolve[x''[t] + x[t] == Sin[t], x[t], t]

  $\ddot{x} + x = \sin t$, general solution.

  DSolve[{x''[t] + x[t] == Sin[t], x[0] == 5, x'[0] == 3}, x[t], t]

  $\ddot{x} + x = \sin t$, with initial conditions $x(0) = 5$ and $\dot{x}(0) = 3$.

  DSolve[{x'[t] == x[t] + 2 y[t], y'[t] == 3 x[t] + 4 y[t]}, {x[t], y[t]}, t]

  Linear system $\dot{x} = x + 2y$, $\dot{y} = 3x + 4y$, general solution.

  The same syntax works in Wolfram Alpha; for instance, you can try out the last example.

Exercises

The problems labelled A1, A2, etc., can be found in the section **Additional problems** just below. The remaining problems (1.1, 1.2, etc.) are from the **course textbook** (Arrowsmith & Place), where they are located at the end of each chapter. Problems which may be more challenging have been marked with an **asterisk** as a warning sign.

- Rehearsal of how to solve ODEs using calculus techniques: A1, A2, 1.4, A3, A4, A5.

  (Those of you who have taken TATA42 might recognize a few of these problems from that course!)

- Sketching solution curves directly from the ODE: 1.11.

- Drawing phase portraits: 1.12, 1.13, A6.

- Phase portraits for parameter-dependent ODEs: 1.14, 1.17*.

  (When the phase portrait changes qualitatively at some particular value of the parameter(s), the system is said to undergo a *bifurcation*.)

- Recovering the ODE from its general solution: A7.

[Table of contents]

Additional problems

A1 Solve using an integrating factor:

(a) $\dot{x} = 2x$.

(b) $\dot{x} = 2x + 7$.

(c) $\dot{x} = 2x + e^{5t}$.

(d) $\dot{x} = 2x + t^2 e^{5t}$.

(e) $\dot{x} = 2x + t^2 e^{2t}$.

(f) $\dot{x} = tx$.

(g) $\dot{x} + 2tx = t$.

(h) $t\dot{x} + 2x = \sin t$, for $t > 0$.

(i) $(1 + t^2)\dot{x} + 2tx = 2t$. Solutions.

A2 Solve using separation of variables:

(a) $\dot{x} = 2x$.

(b) $\dot{x} = tx$.

(c) $\dot{x} = x^2 - 1$.

(d) $t^2\dot{x} = x^2 + 2x + 1$ with $x(-1) = 1$.

(e) $t^2\dot{x} = x^2 + 2x + 1$ with $x(-1) = -1$. Solutions.

A3 Find the general solution $x(t) = x_{\text{hom}}(t) + x_{\text{part}}(t)$ of the following linear constant-coefficient ODEs. (Use the characteristic polynomial to find $x_{\text{hom}}(t)$ and the method of undetermined coefficients to find $x_{\text{part}}(t)$.)

(a) $\dot{x} - 2x = 0$.

(b) $\dot{x} - 2x = 7$.

(c) $\dot{x} - 2x = e^{5t}$.

(d) $\dot{x} - 2x = t^2 e^{5t}$.

(e) $\dot{x} - 2x = t^2 e^{2t}$.

(f) $\ddot{x} + 6\dot{x} + 8x = t + 2e^{-2t}$.

(g) $\ddot{x} + 6\dot{x} + 9x = 0$.

(h) $\ddot{x} + 6\dot{x} + 10x = 2e^{-3t}\cos t$.

(i) $\dddot{x} - x = e^x + \sin x$. Answers.

A4 In the text above, the formula

$$x(t) = \frac{K x_0}{x_0 + (K - x_0)\, e^{-rt}} = \frac{K x_0\, e^{rt}}{K + (e^{rt} - 1)x_0}$$

was given for the solution to the logistic initial value problem,

$$\dot{x} = r\, x\left(1 - \frac{x}{K}\right), \qquad x(0) = x_0.$$

(a) Derive this solution formula using separation of variables.

(b) The same, but by solving the ODE as a Bernoulli equation instead. Solutions.

A5 The *Airy equation* $\ddot{x}(t) = t\,x(t)$ is a second-order ODE with non-constant coefficients. Find the solution which satisfies $x(0) = 1$ and $\dot{x}(0) = 0$, in the form of a power series $x(t) = \sum_{k=0}^{\infty} a_k\, t^k$.

Answer.

A6 Draw the phase portrait for the logistic equation $\dot{x} = r\,x\big(1 - x/K\big)$ with $r > 0$ and $K > 0$. Where should the solution end up if the initial value $x(0) = x_0$ is negative, according to that picture? Now take the explicit solution formula (see problem A4 above) and compute $\lim_{x\to\infty} x(t)$. You should encounter an apparent paradox. Resolve it! Hint.

A7 Find an ODE $\dot{x} = X(t, x)$ whose general solution is given by the one-parameter family of curves $t^2 + x^2 = 2Ct$. (Hint: Differentiate $t^2 + x(t)^2 = 2Ct$ with respect to t, and eliminate the parameter C from the two equations.) Answer.

Lesson 1

The lessons provide some extra time for working on the exercises. This first lesson is also a good opportunity for getting to know each other a little.

Lecture 2. Phase portraits for two-dimensional systems

(Arrowsmith & Place, sections 1.3, 1.4, 1.5.)

Starting from now, you will be expected to watch the video lectures in advance. At the seminar, I will briefly summarize the material, after which there will be time for discussions and for working on the exercises.

We now turn to **two-dimensional** dynamical systems

$$\dot{x}_1 = X_1(x_1, x_2), \qquad \dot{x}_2 = X_2(x_1, x_2),$$

where we assume that the functions X_1 and X_2 are nice enough to guarantee uniqueness and (local) existence of solutions. In vector notation: $\dot{\mathbf{x}}(t) = \mathbf{X}(\mathbf{x}(t))$, or simply $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$. The function $\mathbf{X}$ should be thought of as a **vector field** in phase space: a vector $\mathbf{X}(\mathbf{x})$ is prescribed at each point $\mathbf{x}$. The solution curves for the system are curves $\mathbf{x} = \mathbf{x}(t)$ whose tangent vector $\dot{\mathbf{x}}$ agrees with the prescribed vector $\mathbf{X}(\mathbf{x})$ at each point on the curve. The solution curves are sometimes called **flow lines** of the vector field, and the collection of these curves forms the **phase portrait** of the system. Of course, we cannot draw all these infinitely many curves, so the aim when sketching the phase portrait is to draw a representative *selection* of solution curves – sufficiently many and sufficiently well chosen – so that we get a good geometrical idea of how the system behaves.

- In very simple cases, we can solve the system explicitly. Sometimes we can obtain partial information by methods of calculus.
- But we can also get at least a rough idea of what the phase portrait looks like by direct inspection of the signs of the functions $X_1(x_1, x_2)$ and $X_2(x_1, x_2)$.
- Syntax for drawing phase portraits in Mathematica or Wolfram Alpha:

  StreamPlot[{$X_1(x,y)$**,** $X_2(x,y)$}**,** {**x,** $x_{\min}$**,** $x_{\max}$}**,** {**y,** $y_{\min}$**,** $y_{\max}$ }]

  For example, if the system is $\dot{x} = -1 - x^2 + y$, $\dot{y} = 1 + x - y^2$:

  StreamPlot[{-1-x^2+y, 1+x-y^2}, {x, -3, 3}, {y, -3, 3}]

  (But don't expect an automatic command like this to produce anything nearly as good as the hand-tuned graphics in the textbook.)

- Arrowsmith & Place use the phrase **fixed point** for a point $\mathbf{x}^*$ such that $\mathbf{X}(\mathbf{x}^*) = \mathbf{0}$, but I will usually say **equilibrium point** or just **equilibrium**, and you may come across many other synonyms too: **rest point**, **critical point**, **steady state**, etc.

  The reason for the terminology is of course that $\mathbf{x}(t) = \mathbf{x}^*$ is a constant solution of the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ in this situation. That is, a system starting in the state $\mathbf{x}^*$ must remain in the state $\mathbf{x}^*$ forever. (Provided, of course, that $\mathbf{X}(\mathbf{x})$ is nice enough, so that the solutions are unique!)

- The **evolution operator** or **flow** φ is the function which maps each point $\mathbf{x}$ in phase space to the place where it will be t units of time later if moving as prescribed by the dynamical system. It is denoted by $\varphi_t(\mathbf{x})$ in the book, but it's also common to write $\varphi(t, \mathbf{x})$, since it is simply a function of t and $\mathbf{x}$.

  A fact which isn't mentioned in the textbook is that the flow φ is as "nice" as the vector field $\mathbf{X}$. More precisely, there's a theorem which says that if the system is

  $$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \boldsymbol{\lambda}),$$

  where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ is some vector of parameters, and if $\mathbf{X}$ is of class C^k ($1 \le k \le \infty$) as a function of the variables $(\mathbf{x}, \boldsymbol{\lambda})$, then the flow φ is of class C^k as a function of the variables $(t, \mathbf{x}, \boldsymbol{\lambda})$.

Exercises

As mentioned already on p. 3, problems marked with yellow are **homework problem** to be handed in. (Preferably a few at a time, as soon as you have done them, *not* all problems at once in a big pile at the end!)

- Finding the nullclines for a system (the curves where $\dot{x} = 0$ or $\dot{y} = 0$), and determining the signs of $\dot{x}$ and $\dot{y}$ in between: A8, A9, A10.

- "Connecting the dots": A11.

- Drawing families of parametrized curves and finding the corresponding ODEs: 1.20.

- Exact solution and phase portrait of a linear system: 1.23.

  Clarification: In this problem, you're supposed to express the ODEs in terms the new variables (y_1, y_2), solve them, and draw the phase portrait in the $y_1 y_2$-plane first. Then you map this picture back into the $x_1 x_2$-plane using the change of variables (which is a linear transformation), to obtain the phase portrait for the given system.

- A nonlinear system which can be solved using polar coordinates: 1.25.

  (Can you draw the phase portrait too?)

- Flows: 1.32, 1.36.

  (Hint for 1.32, if you want to solve it from scratch: It's a Bernoulli equation, so divide by x^3 and then let $y = 1/x^2$. Or just use separation of variables and partial fractions directly. But instead of solving, it might be easier to just *verify* that the given flow is correct, by computing $\varphi_0(x_0)$ and $\frac{d}{dt}\varphi_t(x_0)$.)

Additional problems

A8 For each of the functions $f(x, y)$ below, draw the zero level set $\{(x, y) \in \mathbf{R}^2 : f(x, y) = 0\}$, and indicate the regions in the xy-plane where f is positive and where it is negative.

(a) $f(x, y) = (x+1)(y-2)$.

(b) $f(x, y) = y - x^2$.

(c) $f(x, y) = yx^2$.

(d) $f(x, y) = xy - 1$.

(e) $f(x, y) = x^2 - y^2$.

(f) $f(x, y) = x^3 - y^2$.

(g) $f(x, y) = 16 - x^2 - 4y^2$.

A9 For each of the systems $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$ below, do the following:

- First draw a picture with the zero level set for X (the "x-nullcline"), indicating the regions where X is positive/negative with R/L (for "right" and "left", respectively).
- Then draw another picture with the zero level set for Y (the "y-nullcline"), indicating the regions where Y is positive/negative with U/D (for "up" and "down").
- Finally collect all this information in a single picture, i.e., draw the zero level sets for X and Y in the same picture, and mark every region that is formed with RU/RD/LU/LD (for "right and up", etc.). Or, if you prefer, draw arrows: "↗" instead of RU, "↘" instead of RD, etc.
- In this picture, also indicate the system's **equilibrium points**, i.e., the points where the x-nullcline and the y-nullcline intersect (so that $\dot{x}$ and $\dot{y}$ are *both* zero there).

(a) $\dot{x} = 2 + x - y$, $\dot{y} = x^2 + y - 4$. Answer.

(b) $\dot{x} = x(4 - 4x - y)$, $\dot{y} = y(6 - 2x - 3y)$.

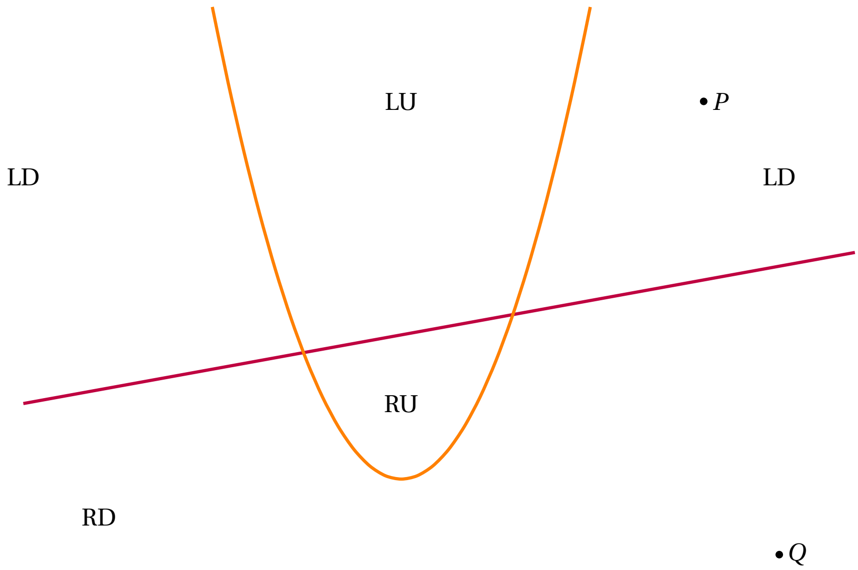
(c) $\dot{x} = y$, $\dot{y} = y(1 - x^2) - x$.

A10 Do the same as in problem A9, but for the system

$$\dot{x} = (x-2)^2 + y - 1, \qquad \dot{y} = (x-1)^2(y+3).$$

A11 Find at least four different solutions to the following task!

Draw a curve (directed, smooth, without self-intersections) which goes from the point P to the point Q in a way that's consistent with the directions given (RU, etc.), Such a curve must have a vertical tangent at the point where it crosses the purple line, and a horizontal tangent at every point where it crosses the orange curve (if it does that), so please be careful to draw it in that way!



[Table of contents]

Lecture 3. Two-dimensional linear systems

(Arrowsmith & Place, sections 2.1, 2.2, 2.3.)

We will spend some time on understanding **linear** dynamical systems in **two dimensions**:

$$\begin{aligned}\dot x_1 &= ax_1 + bx_2 \\ \dot x_2 &= cx_1 + dx_2\end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix}\dot x_1 \\ \dot x_2\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d\end{pmatrix}}_{=A}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}.$$

- Definition of a **simple** linear system: $\det(A) = \left|\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right| = ad - bc$ is **nonzero**. (Equivalently: the eigenvalues of A are nonzero.) This implies that the origin $(x_1, x_2) = (0,0)$ is the **only** equilibrium point. We'll assume simplicity for now, and save non-simple systems for later.
- A linear change of variables $\mathbf{x} = M\mathbf{y}$ turns the system $\dot{\mathbf{x}} = A\mathbf{x}$ into $\dot{\mathbf{y}} = M^{-1}AM\mathbf{y}$.
- "Recipe" for how to choose the columns $\mathbf{m}_1$ and $\mathbf{m}_2$ in the matrix M in order to make $J = M^{-1}AM$ simplify to the **Jordan normal form** of A:
  - (a) Suppose A has two distinct real eigenvalues $\lambda_1 > \lambda_2$. Taking $\mathbf{m}_1$ and $\mathbf{m}_2$ to be the corresponding eigenvectors gives $J = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{smallmatrix}\right)$.
  - (b) Suppose A has a double real eigenvalue λ_0. Then either $A = \left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \lambda_0\end{smallmatrix}\right)$ is already in normal form, or else there is just a one-dimensional eigenspace; in this case let $\mathbf{m}_1$ be an eigenvector and take any vector not parallel to $\mathbf{m}_1$ as our preliminary $\mathbf{m}_2$. This will give a preliminary $J = \left(\begin{smallmatrix} \lambda_0 & C \\ 0 & \lambda_0\end{smallmatrix}\right)$ with some nonzero constant C. Adjust $\mathbf{m}_2$ by dividing it by this constant C; this will give the Jordan form $J = \left(\begin{smallmatrix} \lambda_0 & 1 \\ 0 & \lambda_0\end{smallmatrix}\right)$.
  - (c) Suppose A has a complex-conjugated pair of eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta > 0$. This case is done in a very complicated way in the textbook (p. 39). A much simpler way is to let $\mathbf{a} + i\mathbf{b}$ (with $\mathbf{a}$ and $\mathbf{b}$ real) be a complex eigenvector corresponding to the eigenvalue $\lambda_1 = \alpha + i\beta$. Then $\mathbf{m}_1 = \mathbf{b}$ and $\mathbf{m}_2 = \mathbf{a}$ will work, and directly give the Jordan form $J = \left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha\end{smallmatrix}\right)$. (These vectors are automatically linearly independent – otherwise they would both be *real* eigenvectors of the real matrix A, with a *non-real* eigenvalue λ_1, which is impossible.)
- How to solve the system $\dot{\mathbf{y}} = J\mathbf{y}$ with a matrix in Jordan form, and how to draw its phase portrait.

  In case (a) the phase portrait is a **node** or a **saddle** depending on the signs of the eigenvalues. In case (b) it is a **star node** or an **improper node**. And in case (c) it is a **spiral** (also called **focus**) if $\alpha \neq 0$, or a **centre** if $\alpha = 0$.

  The phase portrait for $\mathbf{x} = M\mathbf{y}$ will be of the same type, only distorted by the linear transformation M. The columns $\mathbf{m}_1$ and $\mathbf{m}_2$ give the **principal directions** of the phase portrait.
- A solution method which isn't mentioned in the book, but is sometimes convenient, is to **rewrite the system as a single second-order ODE** with constant coefficients, as follows. The derivative of $\dot x_1 = ax_1 + bx_2$ is

  $$\ddot x_1 = a\dot x_1 + b\dot x_2.$$

  The second term here, $b\dot x_2$, can be rewritten using the equations from the system:

  $$b\dot x_2 = b(cx_1 + dx_2) = bcx_1 + d(bx_2) = bcx_1 + d(\dot x_1 - ax_1) = d\dot x_1 - (ad - bc)x_1.$$

  So we find that $\ddot x_1 = a\dot x_1 + \big(d\dot x_1 - (ad - bc)x_1\big)$, or in other words

  $$\ddot x_1 - (a + d)\dot x_1 + (ad - bc)x_1 = 0.$$

  This second-order ODE for $x_1(t)$ can now be **solved as in calculus**, with the help of its characteristic polynomial

  $$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc),$$

which (perhaps not surprisingly) coincides with the characteristic polynomial of the system's matrix A, as we can easily check:

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc).$$

And once $x_1(t)$ is known, we also get $x_2(t)$ from $x_2 = (\dot{x}_1 - ax_1)/b$, assuming that $b \neq 0$. (If $b = 0$, the system is easy to solve right away: the first equation $\dot{x}_1 = ax_1 + 0x_2$ gives $x_1(t) = x_1(0)\, e^{at}$; plug this into the second equation and solve the resulting ODE for $x_2(t)$ as in calculus, either using an integrating factor or "homogeneous + particular solution".)

Exercises

- A little reminder of how linear transformations work: 2.1abd.
- Transformation to canonical form: 2.3a, 2.4.
- Solving canonical linear systems: 2.8.
- Drawing phase portraits for canonical linear systems: 2.9.

  There's no answer in the textbook, but remember that you can check your phase portraits using a computer! One case which is particularly easy to get wrong is 2.9d; please verify carefully that your answer agrees with Fig. 2.4 on p. 45 (except that the slope of the dashed nullcline is different).

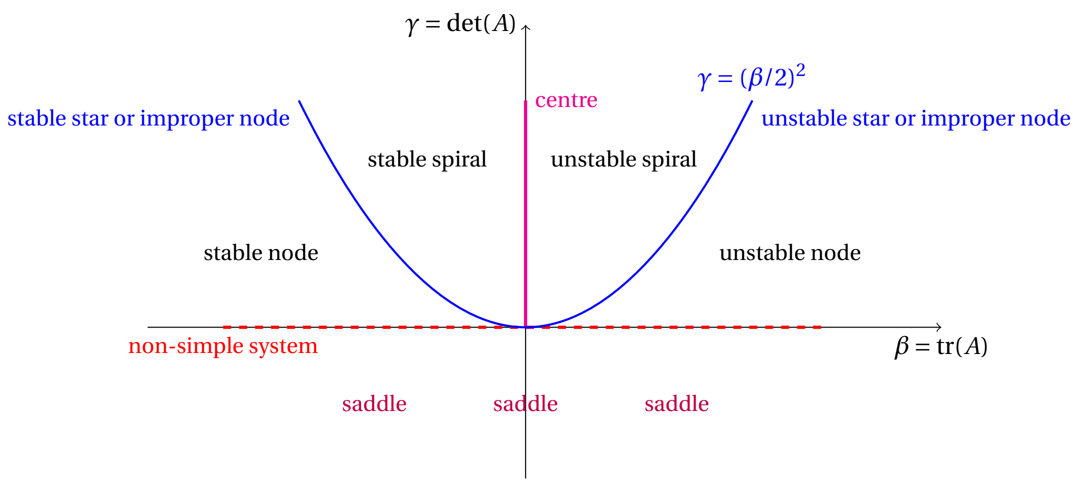
Lecture 4. More about linear systems

(Arrowsmith & Place, sections 2.4, 2.5, 2.6, 2.7.)

- How to tell directly from the coefficients of the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ what the type of the phase portrait for $\dot{\mathbf{x}} = A\mathbf{x}$ is, without computing the eigenvalues: compute the **trace** and the **determinant**,

  $$\beta = \operatorname{tr}(A) = a + d, \qquad \gamma = \det(A) = ad - bc,$$

  and locate the point (β, γ) in the following diagram (cf. Figure 2.7 in the book, p. 47):



  For systems on the parabola $\gamma = (\beta/2)^2$, the type is a star node if the matrix A equals a constant times the identity matrix, otherwise it's an improper node.

  Centres are always stable, and saddles are always unstable (according to the general definitions of "stable equilibrium" and "unstable equilibrium" that will be given later in the course).

- In particular, the zeros of the polynomial $p(z) = z^2 - \beta z + \gamma$ both have *negative real part* if and only if the coefficients satisfy $\beta < 0$ and $\gamma > 0$, i.e., if the point (β, γ) lies in the second quadrant in the diagram above. (This is a special case of the **Routh–Hurwitz criterion**, which determines for a polynomial of arbitrary degree whether all its zeros have negative real part. The name Routh rhymes with "south".)

- The word "type" above refers to a kind of "algebraic type", with two linear systems being of the same type if and only if they are related via a **linear** change of variables. Such changes preserve the eigenvalues, so (for example) an unstable node with $\lambda_1 = 5$ and $\lambda_2 = 3$ cannot be linearly transformed into an unstable node with $\lambda_1 = 43$ and $\lambda_2 = 17$.

  We might want to introduce some looser type of equivalence instead, which will allow (for example) any two unstable nodes to be considered equivalent.

- Classification of **qualitatively equivalent** types of phase portraits for two-dimensional simple linear systems. Under this equivalence relation (**Definition 2.4.1**) there are only four different types:

  (a) Stable node, stable star node, stable improper node, or stable spiral. ("Stable but not centre.")

  (b) Centre.

  (c) Saddle.

  (d) Unstable node, unstable star node, unstable improper node, or unstable spiral. ("Unstable but not saddle.")

  [One subtle detail regarding Definition 2.4.1: They say that two systems are qualitatively equivalent if there is a continuous bijection f which maps the phase portrait of one system onto the phase portrait of the other and preserves the orientation of the trajectories. But how do we know that this relation is *symmetric*? (If system A is equivalent to system B, then we want system B to be equivalent to system A as well.) For this to hold, the inverse f^{-1} must also be a *continuous* bijection. This isn't necessarily true for continuous bijections in general; consider for example the function from the interval $(-\pi, \pi]$ in $\mathbf{R}$ to the unit circle in $\mathbf{R}^2$ given by $f(t) = (\cos t, \sin t)$. But there is a famous and rather difficult theorem by Brouwer ("invariance of domain") which implies that for a continuous bijection between two open sets in $\mathbf{R}^n$ the inverse is automatically continuous, so the definition is actually correct as it stands.]

- Qualitative equivalence is also called **topological equivalence**.

  (A stronger condition is C^k**-equivalence**, where one requires the map f and its inverse f^{-1} to be of class C^k instead of just continuous.)

- As an aside, we may also mention the slightly different notion of **conjugacy**, where one also requires the *time parametrization* of the trajectories to be respected. More precisely, two systems with flows φ and ψ are said to be **topologically conjugate** if there's a continuous map f with continuous inverse f^{-1} such that

$$f \circ \varphi_t = \psi_t \circ f$$

  for all t. That is, following the flow φ of one system for t units of time and then mapping over to the other phase portrait is always the same as first mapping to the other phase portrait and then following *that* flow ψ for t units of time.

  (And the systems are C^k**-conjugate** if f and f^{-1} are of class C^k.)

  For example, a system whose phase portrait consists of concentric circles all traversed with the same period T is *topologically equivalent* to a system whose phase portrait consists of concentric circles traversed with different periods, but the systems are *not topologically conjugate*.

  A conjugacy can also be viewed as a change of variables; if the system with flow φ is described in terms of the variables $\mathbf{x}$, and we make the change of variables $\mathbf{y} = f(\mathbf{x})$, then ψ can be considered as the flow of the same system, just expressed in terms of the new variables $\mathbf{y}$ instead.

- The **exponential function** for square matrices P:

$$e^P = \exp(P) = I + P + \frac{1}{2!}P^2 + \frac{1}{3!}P^3 + \dots$$

- The solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0).$$

  (Here it's important that A is a *constant* matrix, not time-dependent.)

- **Affine** systems $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{h}(t)$, also called **non-homogeneous linear** systems.

  It's easy if $\mathbf{h}$ is time-independent and $A\mathbf{x}_0 = \mathbf{h}$ for some $\mathbf{x}_0$: just set $\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{x}_0$ to get $\dot{\mathbf{y}} = A\mathbf{y}$. But even if this doesn't hold, an affine system can be solved using e^{-At} as an integrating factor.

- Linear systems and Jordan form in n dimensions.

Exercises

- Using the trace–determinant criterion: 2.13.
- More phase portraits: A12.
- Computing matrix exponentials: 2.22, 2.23.
- Affine systems: 2.29abcd, 2.30.
- A three-dimensional linear system: 2.33.
- Solutions corresponding to a 4×4 Jordan block, or two 2×2 Jordan blocks: 2.35.

Additional problems

A12 In each subproblem, draw the phase portrait for the system $\dot{\mathbf{x}} = A\mathbf{x}$ as carefully as you can, but without computing the solution $\mathbf{x}(t)$ explicitly. (Use the trace–determinant criterion or the eigenvalues to determine the type, and compute the the principal directions if there are any. Please indicate the nullclines and the principal directions clearly in your figures.)

(a) $A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & -6 \\ 2 & -1 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 0 & -1 \\ 4 & -4 \end{pmatrix}$. (Pay particular attention to this one – it's easy to get it wrong!)

Lesson 2

Lecture 5. Nonlinear systems, linearization at an equilibrium point

(Arrowsmith & Place, sections 3.1, 3.2, 3.3, 3.4.)

Now back to nonlinear systems $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, in the plane $\mathbf{R}^2$ for simplicity, but the results are valid in $\mathbf{R}^n$ too. We assume that the vector field $\mathbf{X}(\mathbf{x})$ is of class C^1, so that we have existence and uniqueness of solutions; this assumption is also needed for the linearization theorem below to be valid.

- Suppose that $\mathbf{x}^* = (a_1, a_2)$ is an equilibrium point: $X_1(a_1, a_2) = X_2(a_1, a_2) = 0$. Since the functions X_1 and X_2 are assumed to be of class C^1, they are also differentiable, which by definition means that

$$X_1(a_1+h_1, a_2+h_2) = \underbrace{X_1(a_1,a_2)}_{=0} + \frac{\partial X_1}{\partial x_1}(a_1,a_2)\,h_1 + \frac{\partial X_1}{\partial x_2}(a_1,a_2)\,h_2 + \text{remainder},$$
$$X_2(a_1+h_1, a_2+h_2) = \underbrace{X_2(a_1,a_2)}_{=0} + \frac{\partial X_2}{\partial x_1}(a_1,a_2)\,h_1 + \frac{\partial X_2}{\partial x_2}(a_1,a_2)\,h_2 + \text{remainder},$$

where the remainders tend to zero *faster*[1] than $\sqrt{h_1^2+h_2^2}$ as $(h_1,h_2) \to (0,0)$.

- If we discard the remainders, we get a **linear** system for $\mathbf{h}(t) = \mathbf{x}(t) - \mathbf{x}^*$:

$$\dot h_1 = \frac{\partial X_1}{\partial x_1}(a_1,a_2)\,h_1 + \frac{\partial X_1}{\partial x_2}(a_1,a_2)\,h_2,$$
$$\dot h_2 = \frac{\partial X_2}{\partial x_1}(a_1,a_2)\,h_1 + \frac{\partial X_2}{\partial x_2}(a_1,a_2)\,h_2,$$

or in matrix notation,

$$\begin{pmatrix}\dot h_1\\ \dot h_2\end{pmatrix} = \begin{pmatrix}\frac{\partial X_1}{\partial x_1}(a_1,a_2) & \frac{\partial X_1}{\partial x_2}(a_1,a_2)\\ \frac{\partial X_2}{\partial x_1}(a_1,a_2) & \frac{\partial X_2}{\partial x_2}(a_1,a_2)\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix}, \qquad \dot{\mathbf{h}} = \underbrace{\frac{\partial \mathbf{X}}{\partial \mathbf{x}}(\mathbf{x}^*)}_{=A}\,\mathbf{h}.$$

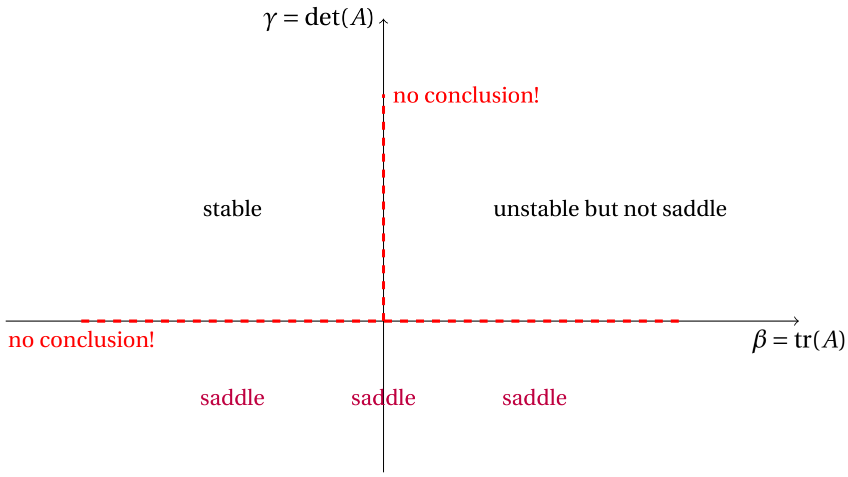
This system is called the **linearization** of the original system at the equilibrium point $\mathbf{x}^*$. Its matrix A is the **Jacobian matrix** of $\mathbf{X}(\mathbf{x})$, *evaluated* at the equilibrium point $\mathbf{x}^*$ (so it's really just a *constant* matrix).

- Our hope is that the linear system $\dot{\mathbf{h}} = A\mathbf{h}$ (which we know how to analyze completely) will tell us something about the behaviour of the original nonlinear system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ *near* the point $\mathbf{x}^*$.

  And this is indeed the case, provided that $\mathbf{x}^*$ is a **hyperbolic**[2] **equilibrium point**, meaning that the Jacobian matrix $A = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}(\mathbf{x}^*)$ has **no eigenvalues on the imaginary axis** in the complex plane.

  Under that condition, **Theorem 3.3.1** (the **linearization theorem**) says that the nonlinear system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is indeed topologically equivalent[3] to its linearization $\dot{\mathbf{h}} = A\mathbf{h}$ in a neighbourhood of $\mathbf{x}^*$.

  In terms of the trace-determinant diagram of $A = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}(\mathbf{x}^*)$:



[1] What this means is that the quotient $R(h_1,h_2)/\sqrt{h_1^2+h_2^2}$ tends to zero as $(h_1,h_2) \to (0,0)$, where $R(h_1,h_2)$ is the remainder.

[2] The word "hyperbolic" is very over-used in mathematics, and it is perhaps not a very good choice in this context, but unfortunately it has become standard terminology. Arrowsmith & Place avoid this word initially, and express the condition by saying that the linearized system should be **simple** ($\lambda = 0$ is not an eigenvalue) and **not a centre** (we don't have $\lambda = 0 \pm ik$ either). But they introduce it a little later, on p. 80.

[3] Actually, it's even topologically conjugate, but the book doesn't introduce that concept.

The linearization theorem is also called the **Hartman–Grobman theorem**, proved independently by Philip Hartman in the U.S.A. and D. M. Grobman in the Soviet Union around 1960. The proof is rather difficult, and way beyond the scope of this course.[4]

(However, a simpler theorem, not dealing with topological equivalence but only with determining *stability* based on linearization, can be proved using Liapunov's theorems that we will learn about next time.)

Exercises

- Linearization: 3.5, 3.6, 3.7, A13.
- An isolated, but non-simple, fixed point: 3.8.
- Non-isolated fixed points: 3.11.

  (To avoid confusion: here "a *line* of fixed points" rather means a *curve*.)
- An explicit example of a topological conjugacy: A14*.
- An example regarding smoothness in the Hartman–Grobman theorem: A15**.

Additional problems

A13 Consider the system

$$\dot{x} = 2(x-y)(x+1), \qquad \dot{y} = x - y^2.$$

(a) Linearize the system at the equilibrium point $(0,0)$ and draw the phase portrait of the linearized system in the hk-plane as carefully as you can. (Determine the type of the equilibrium, indicate the principal directions if there are any, take the nullclines into account, etc.)

(b) Do the same for the other equilibrium point $(1,1)$.

(c) Analyze the signs of $\dot{x}$ and $\dot{y}$ like we done before, for instance in problem A9:

- Draw the x-nullclines $y = x$ and $x = -1$ for the original nonlinear system, and mark the resulting regions in the xy-plane with R/L (right/left).
- Draw the y-nullcline $x = y^2$ in a separate picture, and mark the resulting regions with U/D (up/down).
- Overlay the two pictures in a single picture, and mark the regions with RU/RD/LU/LD.

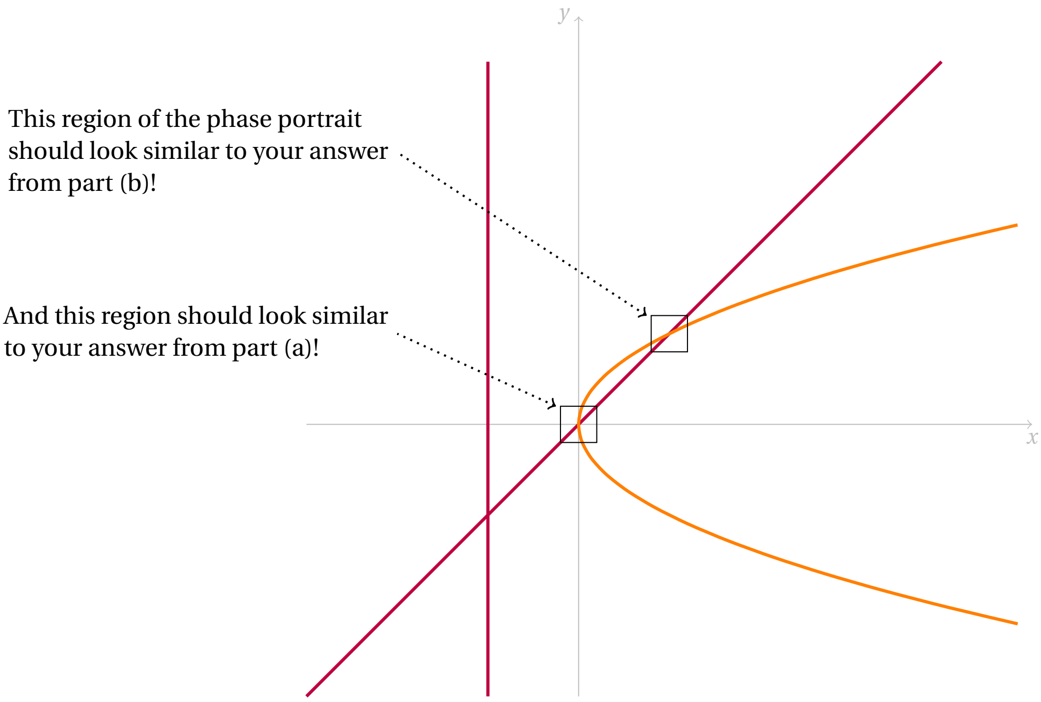
(d) Use the information from the previous parts to draw the system's phase portrait as carefully as you can. Include sufficiently many solution curves to give a reasonably complete picture of the system's global behaviour. In particular, your phase portrait should make it clear what happens inside the unit square $[0,1] \times [0,1]$ and in the vicinity thereof, so don't make your

[4] For two-dimensional systems, a stronger result holds (also proved by Hartman): if the vector field $\mathbf{X}(\mathbf{x})$ is of class C^2, then the nonlinear system is actually C^1-conjugate (not just topologically conjugate) to its linearization, and moreover the derivative of the conjugating map at the origin is the identity. This means, for example, that if the linearization is a saddle, the eigenvectors will tell us the correct incoming and outgoing *directions* of the trajectories of the nonlinear system.

It is not true in three or more dimensions that we can always get C^1-conjugacy. However, it *is* true (as was proved as recently as 2003) that the continuous conjugation map is differentiable *at the origin*, with the derivative there equal to the identity. (The proof assumes that $\mathbf{X}(\mathbf{x})$ is of class C^∞, but the claim is conjectured to be true already for class C^2.)

It's also not true (even in two dimensions) that one can get C^k-conjugacy with $k \geq 2$ by assuming the vector field to be nicer; see problem A15 for a counterexample with a vector field of class C^∞ where one doesn't get more than C^1-conjugacy.

picture too small! Also be careful to make the local phase portraits from parts (a) and (b) fit correctly into the global phase portrait, as indicated in the figure below:



A14 Find a change of variables $(u, v, w) = h(x, y, z)$ which transforms the nonlinear system

$$\dot{x} = -x, \qquad \dot{y} = -y + xz, \qquad \dot{z} = z$$

into the corresponding linearized system at the origin,

$$\dot{u} = -u, \qquad \dot{v} = -v, \qquad \dot{w} = w.$$

Answer.

A15 (a) Show that the change of variables

$$u = x, \qquad v = y + g(x), \qquad \text{where} \quad g(x) = \begin{cases} x^2 \ln|x|, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

converts the nonlinear system $\dot{x} = -x$, $\dot{y} = -2y + x^2$ to its linearization at the origin, $\dot{u} = -u$, $\dot{v} = -2v$.

(b) Show that the function g belongs to the class $C^1(\mathbf{R})$ but not to $C^2(\mathbf{R})$.
Conclude that the above mapping $(u, v) = (x, y + g(x))$ from the xy-plane to the uv-plane is of class $C^1(\mathbf{R}^2)$, but not of class $C^2(\mathbf{R}^2)$, and likewise for the inverse mapping $(x, y) = (u, v - g(u))$.
(Hence the nonlinear system is C^1-conjugate to its linearization. But not C^2-conjugate; see part (d).)

(c) Write down the explicit solutions for both systems in terms of initial data (x_0, y_0) and (u_0, v_0), respectively, and sketch the phase portraits.
(The solution of the nonlinear system can be obtained either by solving the system directly, or by transforming the solution of the linear system using the change of variables above.)

(d) Finally, prove that there is no C^2-conjugacy between the systems (despite the vector field in the nonlinear system being of class C^∞), by filling in the details in the following outline:

- In order to derive a contradiction, assume that there is such a conjugacy $u = A(x, y)$, $v = B(x, y)$, defined (at least) in some neighbourhood Ω of the origin.
(That is, assume that the mapping $f = (A, B)$ is of class C^2, with inverse f^{-1} of class C^2, and that it relates the flows of the two systems as in the definition of conjugacy on p. 15).
- Explain why such a mapping must have nonzero Jacobian determinant everywhere. (Hence, in particular, at the origin.)
- Explain why the functions A and B must satisfy the following functional equations for all $(x, y) \in \Omega$ and all t sufficiently close to 0 (so that both sides are defined):

$$A(x, y)\,e^{-t} = A\big(x\,e^{-t}, (y + tx^2)\,e^{-2t}\big),$$
$$B(x, y)\,e^{-2t} = B\big(x\,e^{-t}, (y + tx^2)\,e^{-2t}\big).$$

- Using the assumption that A and B are of class C^2, apply the operator $(\partial/\partial x)^2$ to these identities, and insert $(x, y) = (0, 0)$ afterwards, to get

$$e^{-t} A_{xx}(0,0) = 2te^{-t} A_y(0,0) + e^{-2t} A_{xx}(0,0),$$
$$e^{-2t} B_{xx}(0,0) = 2te^{-t} B_y(0,0) + e^{-2t} B_{xx}(0,0).$$

  Conclude, since these identities have to hold for all t in some interval, that $A_y(0,0) = 0$ and $B_y(0,0) = 0$ (and $A_{xx}(0,0) = 0$, but that's not so relevant here).
- This means that the Jacobian determinant is zero at the origin, which is the desired contradiction.

Lecture 6. Stability theorems

(Arrowsmith & Place, sections 3.5, 3.6, 3.7.)

- **Definitions 3.5.1–4**: The definition of what it means for an equilibrium point $\mathbf{x}^*$ to be **(Liapunov) stable**: for every neighbourhood U of $\mathbf{x}^*$ there is a neighbourhood $U' \subseteq U$ of $\mathbf{x}^*$ such that trajectories starting in U' cannot leave U.

  Some stable equilibria are **asymptotically stable**, meaning that (in addition to the above requirement for stability) there is a neighbourhood N of $\mathbf{x}^*$ such that every trajectory starting in N converges to $\mathbf{x}^*$ as $t \to \infty$.

  And those which are stable but not asymptotically stable are called **neutrally stable**. So there are exactly those two types of stable equilibria: asymptotically stable and neutrally stable.

  Unstable equilibrium simply means an equilibrium which is not stable.

- Russian names can be transliterated into the Latin alphabet in many ways. Here I'm writing **Liapunov** like in the textbook, but a very common alternative in English is **Lyapunov**, and one may also come across **Ljapunow**, **Liapounoff**, and so on. Anyway, it's pronounced with the stress on the last syllable: "-OFF".

- **Theorem 3.5.1** is **Liapunov's stability theorem** (1892).

  This theorem is useful for showing stability in situations where linearization is inconclusive. Even more importantly, it also provides a **domain of stability**, which is the textbook's terminology (although they don't really define it precisely) for a neighbourhood N of an asymptotically stable equilibrium $\mathbf{x}^*$ such that any solution which starts in N stays in N and converges to $\mathbf{x}^*$ as $t \to \infty$. (From linearization we can only say that if all eigenvalues have negative real part, then there is *some* domain of stability, but we don't get any clue about the *size* of that domain.)

However, Arrowsmith & Place don't state exactly how to find a domain of stability, and their proof of the theorem is also rather unclear. I will try to give more precise statements here.

First let us fix some terminology.

Definition. Let I be a proper[1] interval in $\mathbf{R}$, and let f be a real-valued function whose domain of definition contains I. The function f is said to be **strictly decreasing** on I if

$$f(t_1) > f(t_2)$$

whenever $t_1 \in I$, $t_2 \in I$ and $t_1 < t_2$. It is **weakly decreasing** on I if

$$f(t_1) \geq f(t_2)$$

whenever $t_1 \in I$, $t_2 \in I$ and $t_1 < t_2$.

Remark. Any function which is strictly decreasing is weakly decreasing as well.

Remark. In English it is more common to say **decreasing** and **non-increasing** instead of **strictly decreasing** and **weakly decreasing**, but I have chosen the latter option here to reduce the risk of confusion with the usual terminology in Swedish, which is **strängt/strikt avtagande** and **avtagande**, respectively. (And similarly in German and French.)

From now on we consider some fixed dynamical system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ in $\mathbf{R}^n$, where the vector field $\mathbf{X}$ is defined in some open set $S \subseteq \mathbf{R}^n$, and we assume that $V: \Omega \to \mathbf{R}$ is a differentiable function defined on some open set $\Omega \subseteq S$. (And of course we assume that Ω and S are not the empty set, since that wouldn't be very interesting.)

Definition. The function $\dot{V}: \Omega \to \mathbf{R}$ is the dot product of the gradient ∇V and the vector field $\mathbf{X}$:

$$\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{X}(\mathbf{x}), \qquad \mathbf{x} \in \Omega.$$

Theorem. Suppose $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \Omega$. Then, for any solution $\mathbf{x}(t)$ of the system which stays in the set Ω during some nonempty open time interval I, the function

$$f(t) = V(\mathbf{x}(t)), \qquad t \in I$$

is strictly decreasing on I. If instead $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$, then f is weakly decreasing on I. And if $\dot{V}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega$, then f is constant on I.

Proof. The chain rule gives

$$\begin{aligned} f'(t) &= \nabla V(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) \\ &= \nabla V(\mathbf{x}(t)) \cdot \mathbf{X}(\mathbf{x}(t)) = \dot{V}(\mathbf{x}(t)), \qquad t \in I. \end{aligned}$$

So if we know that $\dot{V} < 0$ everywhere in Ω, and that $\mathbf{x}(t)$ stays in Ω for $t \in I$, then we have $f'(t) < 0$ for $t \in I$, which implies that f is strictly decreasing on I (by a very basic calculus theorem). Similarly if $\dot{V} \leq 0$ or $\dot{V} = 0$. □

Theorem (Liapunov's stability theorem, weak version)**.** Let $\mathbf{x}^*$ be an equilibrium point of the dynamical system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in S \subseteq \mathbf{R}^n$. Suppose that there is a **weak Liapunov function**, i.e., a differentiable function $V: \Omega \to \mathbf{R}$ defined on some open set $\Omega \subseteq S$ containing $\mathbf{x}^*$ and satisfying the conditions

1. $V(\mathbf{x}^*) = 0$, and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega \setminus \{\mathbf{x}^*\}$,
2. $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$.

[1] A **proper interval** is an interval (in $\mathbf{R}$) which contains infinitely many points, as opposed to the *degenerate* intervals $[a, a] = \{a\}$ and $\emptyset = \{\}$.

Then the equilibrium $\mathbf{x}^*$ is **stable**.

Remark. As the theorem is formulated, the function $V : \Omega \to \mathbf{R}$ has *exactly* the set Ω as its domain of definition. Usually we have some nice function V (typically a polynomial) which is defined *everywhere* to begin with. What we actually do then is that we compute $\dot{V}$, use that to locate some open set Ω where the assumptions of the theorem are fulfilled, and then apply the theorem with V equal to the *restriction* of the original function V to the set Ω.

Outline of proof. If U is any neighbourhood of $\mathbf{x}^*$, let $B \subset U \cap \Omega$ be a closed ball centered at $\mathbf{x}^*$ and set

$$U' = \{\mathbf{x} \in B : V(\mathbf{x}) < \alpha\},$$

where $\alpha > 0$ is the minimum of V on the boundary sphere ∂B. Then $U' \subset B \subset U$, and U' is a neighbourhood of $\mathbf{x}^*$ such that trajectories starting in U' can't leave U (in fact, they can't even leave U'). □

Detailed proof. Let U be an arbitrary neighbourhood of $\mathbf{x}^*$. To prove stability, we need to find another neighbourhood U' such that solutions starting in U' will never leave U. To find U' we begin by taking a closed ball

$$B = \overline{B(\mathbf{x}^*, \varepsilon)} = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x} - \mathbf{x}^*| \le \varepsilon\}$$

centered at $\mathbf{x}^*$, with radius $\varepsilon > 0$ small enough for B to be contained inside both U and Ω. (This is possible since U and Ω are neighbourhoods of $\mathbf{x}^*$.) The boundary

$$\partial B = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x} - \mathbf{x}^*| = \varepsilon\}$$

is a sphere of radius ε centered at $\mathbf{x}^*$. This sphere is a compact set (closed and bounded), and V is continuous by assumption, so according to the extreme value theorem V has a smallest value on ∂B:

$$\alpha = \min_{\mathbf{x} \in \partial B} V(\mathbf{x}).$$

In other words, there is a point $\mathbf{x}_0 \in \partial B$ such that

$$\alpha = V(\mathbf{x}_0) \le V(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \partial B.$$

Since V is positive definite on Ω (i.e., satisfies condition 1 in the statement of the theorem) and we have chosen B small enough to be a subset of Ω, we have $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \partial B$, in particular

$$\alpha = V(\mathbf{x}_0) > 0.$$

Now set

$$U' = \{\mathbf{x} \in B : V(\mathbf{x}) < \alpha\}.$$

Then U' contains $\mathbf{x}^*$, since $V(\mathbf{x}^*) = 0 < \alpha$. And U' is an open set, since V is continuous.[2] In other words, U' is an open neighbourhood of $\mathbf{x}^*$. Moreover, a trajectory $\mathbf{x}(t)$ starting in U' (at $t = 0$, say) can't leave U. Here's why: to leave U, the trajectory would have to leave B to begin with (since $U' \subset B \subset U$), and it's a continuous curve so it would have to intersect the boundary sphere ∂B in order to get out. But $f(t) = V(\mathbf{x}(t))$ is a weakly decreasing function of t as long as $\mathbf{x}(t)$ stays in B (since $B \subset \Omega$, and $\dot{V} \le 0$ in Ω by assumption). Since we start in U', we have $f(0) < \alpha$, and hence $f(t) < \alpha$ for $t \ge 0$. So it's impossible for the trajectory to reach ∂B, since that would mean $f(t) \ge \alpha$ for some $t > 0$. (In fact, the trajectory can't even leave U' – as soon as it did, it would mean that $f(t) \ge \alpha$.)

(One more technical detail: since the trajectory stays inside the compact set B, it must exist for all $t \ge 0$; there can't be any "blowup in finite time". We haven't proved that theorem in this course, so we'll just accept this fact on faith here.) □

[2] Take any $\mathbf{x} \in U'$, or in other words any $\mathbf{x} \in B$ with $V(\mathbf{x}) < \alpha$. Then actually $\mathbf{x}$ is in the interior of B, since $V \ge \alpha$ on the boundary ∂B. Continuity of V at $\mathbf{x}$ means that there is an open ball $B_2 = B(\mathbf{x}, \delta)$ where $V < \alpha$, and this ball must also be contained in the interior of B, for the same reason. So $B_2 \subseteq U'$. Thus any $\mathbf{x} \in U'$ has an open neighbourhood contained in U', and this is exactly what it means for U' to be open.

Theorem (Liapunov's stability theorem, strong version)**.** Let $\mathbf{x}^*$ be an equilibrium point of the dynamical system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in S \subseteq \mathbf{R}^n$. Suppose that there is a **strong Liapunov function**, i.e., a differentiable function $V : \Omega \to \mathbf{R}$ defined on some open set $\Omega \subseteq S$ containing $\mathbf{x}^*$ and satisfying the conditions

1. $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega \setminus \{\mathbf{x}^*\}$,
2. $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \Omega \setminus \{\mathbf{x}^*\}$.

Then the equilibrium $\mathbf{x}^*$ is **asymptotically stable**. In fact, for any closed ball $B = \overline{B(\mathbf{x}^*, r)}$ contained in Ω, the set

$$N = \{\mathbf{x} \in B : V(\mathbf{x}) < \alpha\}, \quad \text{where } \alpha = \min_{\mathbf{x} \in \partial B} V(\mathbf{x}),$$

is a **domain of stability**: solutions starting in N stay in N, and converge to $\mathbf{x}^*$ as $t \to \infty$.

Remark. If $\Omega = \mathbf{R}^n$ and if the additional condition

$$V(\mathbf{x}) \to \infty \quad \text{as} \quad |\mathbf{x}| \to \infty$$

holds, then for *any* given point $\mathbf{x}_0 \in \mathbf{R}^n$ it is true that

$$\min_{\mathbf{x} \in \partial B} V(\mathbf{x}) > V(\mathbf{x}_0)$$

if we take B large enough. This means that $\mathbf{x}_0 \in N$ for that choice of B, causing the trajectory starting at $\mathbf{x}_0$ to converge to $\mathbf{x}^*$. So in this case $\mathbf{x}^*$ is stable and *every* trajectory of the system converges to $\mathbf{x}^*$, which is expressed by saying that $\mathbf{x}^*$ is **globally asymptotically stable**.

Remark. If one studies the proof, it should be clear that the set B in the definition of N doesn't really have to be precisely a closed ball, just something which is topologically equivalent to a ball. For example, if our Liapunov function is $V(x, y) = x^6 + y^4$, then for $k > 0$ the sublevel set $B = \{x^6 + y^4 \le k\}$ is sufficiently "ball-like" for the proof to work: no continuous curve can pass from the interior to the exterior without crossing the closed level curve $\partial B = \{x^6 + y^4 = k\}$. The same arguments as in the proof then show that trajectories can't leave the set $N = \{\mathbf{x} \in B : V(x, y) < k\} = \{x^6 + y^4 < k\}$, so provided that B is contained in the region Ω, N is a domain of stability for the equilibrium $(0, 0)$.

Outline of proof. Stability follows from the weak version of Liapunov's theorem. Any trajectory starting in N must stay in N, and along such a trajectory the function V decreases strictly towards some limit $L \ge 0$. But $L > 0$ would contradict the continuity of the flow φ_t, so $L = 0$, which in turn implies that the trajectory converges to $\mathbf{x}^*$ (the only point where $V = 0$). □

Detailed proof. Stability follows from the weak version of Liapunov's theorem. As in the proof of that theorem, we see that N (as defined above) is an open neighbourhood of $\mathbf{x}^*$, and that any trajectory $\mathbf{x}(t)$ starting in N stays in N and is defined for all $t \ge 0$. For the constant solution $\mathbf{x}(t) = \mathbf{x}^*$ there is nothing to prove – of course it converges to $\mathbf{x}^*$! So suppose $\mathbf{x}(t)$ is some *other* solution starting in N. Then, by the assumption $\dot{V} < 0$, $V(\mathbf{x}(t))$ is a strictly decreasing function of t on the interval $t \ge 0$, and it's bounded below (since $V \ge 0$), so it has a limit $L \ge 0$ as $t \to \infty$.

We want to show that $L = 0$, so assume $L > 0$ in order to get a contradiction. Take any sequence of positive numbers $t_n \nearrow \infty$; then $\mathbf{x}_n = \mathbf{x}(t_n)$ is a sequence of points in the compact set B. According to the Bolzano–Weierstrass theorem (a standard theorem about compact sets in $\mathbf{R}^n$), this sequence of points must have a convergent subsequence, i.e., there is a point $\mathbf{y} \in B$ and an integer sequence $n_k \nearrow \infty$ such that $\mathbf{x}_{n_k} \to \mathbf{y}$ as $k \to \infty$. Since V is continuous, we can move the limit outside V and obtain

$$V(\mathbf{y}) = V\left(\lim_{k \to \infty} \mathbf{x}_{n_k}\right) = \lim_{k \to \infty} V(\mathbf{x}_{n_k}) = L.$$

We are assuming $L > 0$, which means that $\mathbf{y} \ne \mathbf{x}^*$ (since V is positive definite), and V will thus continue to decrease *strictly* along the trajectory starting at $\mathbf{y}$. So the flow φ_1, for example (or φ_t

for any fixed $t > 0$), will map $\mathbf{y}$ to a point where $V < L$. But φ_1 is a continuous function, so it will also map all sufficiently nearby points $\mathbf{x}_{n_k}$ to points where $V < L$:

$$V(\varphi_1(\mathbf{x}_{n_k})) = V(\mathbf{x}(1 + t_{n_k})) < L, \qquad \text{for all sufficiently large } k.$$

But we know that $V(\mathbf{x}(t)) > L$ for all $t \geq 0$, since $V(\mathbf{x}(t))$ is *decreasing* towards the limit L. This contradiction shows that the assumption $L > 0$ must have been incorrect. Hence $L = 0$.

Now we know that $V(\mathbf{x}(t)) \searrow L = 0$ as $t \to \infty$. It remains to show that this implies $\mathbf{x}(t) \to \mathbf{x}^*$, i.e., that for any $\varepsilon > 0$ there is a time τ such that $\mathbf{x}(t) \in B_\varepsilon$ for all $t > \tau$, where $B_\varepsilon = B(\mathbf{x}^*, \varepsilon)$ is the open ball of radius ε centered at $\mathbf{x}^*$. We may assume that $0 < \varepsilon < r$, where r is the radius of the closed ball B. Then

$$B \setminus B_\varepsilon = \overline{B(\mathbf{x}^*, r)} \setminus B(\mathbf{x}^*, \varepsilon)$$

is a compact nonempty set, so the continuous function V has a smallest value β on this set (and $\beta > 0$ since V is positive definite). What this means is that if $\mathbf{x} \in B$ and $V(\mathbf{x}) < \beta$, then $\mathbf{x} \in B_\varepsilon$. But we have $\mathbf{x}(t) \in B$ for all $t \geq 0$, and since $V(\mathbf{x}(t)) \searrow 0$ as $t \to \infty$ there is a τ such that $V(\mathbf{x}(t)) < \beta$ for $t > \tau$. Consequently $\mathbf{x}(t) \in B_\varepsilon$ for $t > \tau$, as desired. □

- The very useful **Theorem 3.5.2** is an improvement of Liapunov's theorem which is due to LaSalle (1960). It allows us to conclude *asymptotic* stability using only a *weak* Liapunov function, provided an additional condition is satisfied. The proof is not given in the book, but it is a consequence of something called **LaSalle's invariance principle** (see the next lecture).

Theorem (LaSalle's stability theorem)**.** Let $\mathbf{x}^*$ be an equilibrium point of the dynamical system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in S \subseteq \mathbf{R}^n$. Suppose that there is a **weak Liapunov function** $V : \Omega \to \mathbf{R}$ on some open set $\Omega \subseteq S$ containing $\mathbf{x}^*$, and in addition suppose that the set

$$\left\{\mathbf{x} \in \Omega : \dot{V}(\mathbf{x}) = 0\right\}$$

contains no complete trajectory except the constant solution $\mathbf{x}^*$.

Then the equilibrium $\mathbf{x}^*$ is **asymptotically stable**, and N (defined as in the strong version of Liapunov's theorem) is a **domain of stability**.

(And if $\Omega = \mathbf{R}^n$ and $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, then $\mathbf{x}^*$ is **globally asymptotically stable**.)

Proof. See the next lecture. □

- Here is a somewhat subtle point concerning the above theorems and weak Liapunov functions. If we have a function V which is everywhere defined and nice (continuously differentiable), then the set $\Omega_1 = \left\{\mathbf{x} \in \mathbf{R}^n : \dot{V}(\mathbf{x}) \leq 0\right\}$ will be *closed*. But the theorems, as they are formulated above, require us to restrict V to an *open* set Ω. So we have to shrink Ω_1 "by hand" to get an *open* set Ω where $\dot{V} \leq 0$ holds. The purpose of this, as well as all the business with closed balls contained inside Ω, is to avoid accidentally making plausible-sounding claims which are actually false. We know that V is weakly decreasing along trajectories, but only as long as they stay in Ω, so we need to take some precautions to prevent the trajectories from sneaking out of Ω!

Example. If the system is $\dot{x} = y$, $\dot{y} = -x - y(1 - x^2)$, and if $V(x, y) = x^2 + y^2$, then $\dot{V} = -2y^2(1 - x^2)$. Thus, the set $\Omega_1 = \{\dot{V} \leq 0\}$ is the union of the closed strip $-1 \leq x \leq 1$ and the line $y = 0$. So any trajectory will be moving closer to the origin (or at least not further away from it) as long as it is inside the strip, but trajectories for $y > 3$ (or so) will enter the strip from the left, *leave it again on the right* (a little further down), and then go off steeply upwards towards infinity instead of converging towards the equilibrium $(0, 0)$. So for example, a set like $\{(x, y) \in \Omega_1 : V(x, y) \leq 100\}$ is not forward invariant despite V being weakly decreasing on trajectories in Ω_1! But we can take Ω to be the open strip $-1 < x < 1$, let B be any closed ball $x^2 + y^2 \leq k$ with $0 < k < 1$ so that it fits inside Ω, and then the set N given by $x^2 + y^2 < k$ will be a domain of attraction by LaSalle's theorem, since there are no trajectories contained in the line $y = 0$ except the equilibrium solution

$(x(t), y(t)) = (0,0)$. And since this is true for any $0 < k < 1$, in fact the open unit disk $x^2 + y^2 < 1$ is a domain of attraction.[3]

- **Theorem 3.5.3** can be formulated as follows:

  Theorem (Liapunov's instability theorem)**.** Let $\mathbf{x}^*$ be an equilibrium point of the dynamical system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in S \subseteq \mathbf{R}^n$. Suppose there is a differentiable function $V : \Omega \to \mathbf{R}$ defined on some open set $\Omega \subseteq S$ containing $\mathbf{x}^*$ and satisfying the conditions

  1. $V(\mathbf{x}^*)$ is not a local maximum,
  2. $\dot{V}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega \setminus \{\mathbf{x}^*\}$.

  Then the equilibrium $\mathbf{x}^*$ is **unstable**.

  Idea of proof. For any closed ball $N = \overline{B(\mathbf{x}^*, r)} \subset \Omega$, there is a point $\mathbf{x}_0$ in the interior of N such that $V(\mathbf{x}_0) > V(\mathbf{x}^*)$, and it is shown that the trajectory starting at such a point $\mathbf{x}_0$ must leave N. □

- We have relied upon the rather deep Hartman–Grobman theorem to show that an equilibrium is asymptotically stable if the linearization there is asymptotically stable. This fact can be proved more directly using Liapunov's stability theorem. The simplest case is when the Jacobian matrix $A = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}(\mathbf{x}^*)$ has distinct real eigenvalues (assumed negative, in order for $d\mathbf{h}/dt = A\mathbf{h}$ to be asymptotically stable):
  $$0 > \lambda_1 > \cdots > \lambda_n.$$
  Make the usual linear change of coordinates $\mathbf{x} = M\mathbf{y}$, where the columns of M form a basis of eigenvectors of A. In terms of these coordinates, we have a system $\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y})$ with an equilibrium $\mathbf{y}^*$ where the Jacobian is diagonal:
  $$J = \frac{\partial \mathbf{Y}}{\partial \mathbf{y}}(\mathbf{y}^*) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$
  So $\mathbf{k} = \mathbf{y} - \mathbf{y}^*$ satisfies
  $$\dot{\mathbf{k}} = J\mathbf{k} + \text{remainder},$$
  where the remainder tends to zero faster than $|\mathbf{k}|$. It's not too difficult to check that $V(\mathbf{k}) = \sum k_i^2$ is a strict Liapunov function for this system in a neighbourhood of $\mathbf{k} = \mathbf{0}$, which proves asymptotic stability. With repeated and/or non-real eigenvalues things are a bit more complicated, but if all eigenvalues have negative real part, one can find a strong Liapunov function in the form of a sum of squares in those cases too.

  Similarly, one can use Liapunov's instability theorem to prove that if some eigenvalue has positive real part, then the equilibrium is unstable.

- The **flow box theorem**. Global phase portraits.

- **First integrals**, also known as **constants of motion**, **integrals of motion**, **conserved quantities**, **invariants**, etc.

  (Can sometimes be found by writing $dy/dx = \dot{y}/\dot{x} = Y(x,y)/X(x,y)$ and solving the resulting ODE for $y = y(x)$.)

- If $H(q,p)$ is any C^1 function, then the **Hamiltonian system**
  $$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H(q,p)$$
  automatically has H as a first integral. A fundamental fact in mechanics is that the Hamiltonian system generated by $H(q,p) = \frac{1}{2}p^2 + V(q)$ is equivalent to the Newton-type equation $\ddot{q} = -V'(q)$. (This works also in higher dimensions, with vectors $\mathbf{p}$ and $\mathbf{q}$ instead.)

[3] We can actually do yet a little better: the closed unit disk $x^2 + y^2 \le 1$ is forward invariant since the open unit disk is; this follows from the continuity of the flow, since if φ_t (for some $t > 0$) would map some point on the unit circle to a point outside the circle, then by continuity it would also have to map some nearby point inside the circle out of the circle, which we know it doesn't. So the closed unit disk is compact and forward invariant, and then we can try applying LaSalle's invariance principle (see next lecture) to it. This turns out to be successful (exercise), so actually the *closed* unit disk is a domain of attraction.

Exercises

- $V(x,y) = x^2 + y^2$ as a strong Liapunov function: 3.13abe, 3.14abe.
- $V(x,y) = x^2 + y^2$ as a weak Liapunov function: 3.15, A16.
- Less obvious strong Liapunov functions: A17, A18*.
- Less obvious weak Liapunov functions: A19, 3.19b, A20.
- Using positive definite quadratic forms as Liapunov functions: 3.18*.

  Please note that there is a **sign error** in the given ODE for x_2; the system is supposed to be

  $$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + (x_1 + 2x_2)(x_2^2 - 1).$$

  Also note that the answer given in the book is not the only possible one.[4]
- More about quadratic forms, for *linear* systems: A21*.
- An example showing that the assumption $V(\mathbf{x}) \to \infty$ is important in order to get *global* asymptotic stability: A22.
- An example[5] illustrating why the requirement about stability in the definition of asymptotic stability is necessary: A23.
- Showing instability: 3.22.

  (First do it as the book suggests, using Liapunov's instability theorem. Then find a much simpler way of showing that the origin is unstable, by thinking about what the phase portrait looks like!)
- And one more about Liapunov's instability theorem: A24*.
- Illustration of an explicit transformation which straightens out a nonlinear vector field: 3.24.
- Constants of motion (first integrals): 3.28ab, 3.29*, A25*.

  In 3.28b, you can do better than the answer in the book, and find a constant of motion which is actually defined for *all* x_1 and x_2!

  In 3.29, you should do much better than the answer in the book, which is actually wrong. A correct constant of motion is $F(x_1,x_2) = x_1(x_1^2 - x_2)/(x_1^2 + x_2)^2$. The level curves of this function are not easy to plot by hand, but you can of course do it on the computer if you want to see what the phase portrait looks like.

Additional problems

A16 Draw the phase portrait (first by hand, and then on the computer for verification) for the system

$$\dot{x} = y, \qquad \dot{y} = -x - y(1 - x^2)$$

from the example above (on p. 24). In the same picture, draw some level sets of the weak Liapunov function $V(x,y) = x^2 + y^2$ and indicate the sets where $\dot{V} < 0$ and $\dot{V} = 0$. Make sure that you understand why the strip $|x| < 1$ is *not* a domain of stability!

[4] Since

$$-\tfrac{1}{2}\dot{V} = x_1^2 + x_2^2 + (3-a)x_1x_2 + (1 - x_2^2)(bx_1 + cx_2)(x_1 + 2x_2),$$

it's quite natural to pick $b = 1$ and $c = 2$ to get a term $(1 - x_2^2)(x_1 + 2x_2)^2$ whose sign we have control over, but then we can take any $a \in [1,5]$ to make $\dot{V}$ negative definite in the strip $|x_2| < 1$ (and note also that all these values of a satisfy the conditions $a > 0$ and $2a = ac > b^2 = 1$ for making V positive definite). You might want to draw the phase portrait and your domain of stability on the computer! And why not try this for different values of a? If you take the *union* of the different domains of stability for $1 \le a \le 5$ you get a bigger (=better) domain of stability!]

[5] The system (3.33) in the book is another such example, but that one is much more difficult to analyze rigorously (Exercise 3.12); for details about this, see Section 40 of the book *Stability of Motion* by W. Hahn (Springer, 1967).

A17 Show that the origin is a globally asymptotically stable equilibrium point of the system

$$\dot{x} = -x + 6y^3 - 3y^4, \qquad \dot{y} = -x - y + \tfrac{1}{2}xy.$$

(Hint: Look for a strong Liapunov function of the form $x^2 + cy^k$.) Solution.

A18 Show that the origin is an asymptotically stable equilibrium for the system

$$\dot{x} = 2y + (x^2 + y^2 - 1)x, \qquad \dot{y} = -4(x - x^3) + (x^2 + y^2 - 1)y,$$

and find a domain of stability. (Hint: $V(x, y) = 2x^2 + y^2 - x^4$.) Solution.

A19 Show that the origin is a stable equilibrium of the system

$$\dot{x} = -2y, \qquad \dot{y} = 2x + x^2 - y^3.$$

(Hint: $V(x, y) = x^2 + y^2 + \frac{1}{3}x^3$.) Is it asymptotically stable? If so, is it globally asymptotically stable? Solution.

A20 Show that $V(x, y) = x^2 + y^2 - x^2y^2$ is a weak Liapunov function for the system

$$\dot{x} = -y(1 - x^2), \qquad \dot{y} = (x - y^3)(1 - y^2).$$

Use LaSalle's theorem to show that the origin is asymptotically stable, and that the open unit square

$$D = \{(x, y) : |x| < 1, |y| < 1\}$$

is a domain of stability.

A21 Here is an algorithm for finding Liapunov functions for *linear* $n \times n$ systems $d\mathbf{x}/dt = A\mathbf{x}$ such that all eigenvalues of A have negative real part (so that the origin is asymptotically stable):

> Take an arbitrary positive definite $n \times n$ matrix Q (symmetric), and solve for the symmetric $n \times n$ matrix P in the **Liapunov equation**
>
> $$A^T P + PA = -Q.$$
>
> Then the matrix P will be positive definite,[6] and
>
> $$V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$$
>
> will be a strong Liapunov function with $\dot{V}(\mathbf{x}) = -\mathbf{x}^T Q \mathbf{x}$.

Try this out on the system

$$\dot{x} = x + y, \qquad \dot{y} = -5x - 2y,$$

with the positive definite matrix $Q = \operatorname{diag}(2, 4)$. (If you're lazy, you can get some help from Wolfram Alpha.) Verify that the function V that you obtain really is a strong Liapunov function for this system! Answer.

A22 Show that

$$V(x, y) = \frac{x^2}{1 + x^2} + y^2$$

is a global strong Liapunov function for the system

$$\dot{x} = -x(1 - 3x^2y^2), \qquad \dot{y} = -y(1 + x^2y^2),$$

but that the origin is **not** globally asymptotically stable (although it is locally asymptotically stable). (Note that $V(x, y)$ does *not* satisfy the condition that $V(x, y) \to \infty$ as $\sqrt{x^2 + y^2} \to \infty$. What do the level sets of V look like, in particular the level set $V = 1$?) Hints.

[6] If you do this for a system where the origin is not asymptotically stable, then the matrix P that you get will not be positive definite.

A23 Consider the system

$$\dot{x} = x\left(1 - x^2 - y^2\right) - y\left(x^2 + y^2 - x\sqrt{x^2 + y^2}\right),$$
$$\dot{y} = y\left(1 - x^2 - y^2\right) + x\left(x^2 + y^2 - x\sqrt{x^2 + y^2}\right).$$

Rewrite this system in polar coordinates.

Use this to draw the phase portrait. Deduce that, with the exception of the equilibrium solution $(x, y) = (0, 0)$, all solutions $(x(t), y(t))$ approach the point $(x, y) = (1, 0)$ as $t \to \infty$, but $(1, 0)$ is still an **unstable** equilibrium. Answer.

A24 Let $V(x, y) = -(y - x^3)(y - x^5)$, and consider the system

$$\dot{x} = \frac{\partial V}{\partial x} = -8x^7 + 3x^2 y + 5x^4 y, \qquad \dot{y} = \frac{\partial V}{\partial y} = x^3 + x^5 - 2y.$$

(a) Have a look at the phase portrait on the computer (Wolfram Alpha link). Can you tell from the graphics whether the origin is stable or unstable?

(b) Prove that the origin is in fact unstable, using Liapunov's instability theorem with the given function V.

A25 Find a constant of motion for the system in problem 3.22, and use this to draw the phase portrait accurately. Solution.

Lecture 7. Limit sets

(Arrowsmith & Place, sections 3.8, 3.9.)

- **Definition 3.8.1** and the paragraph just below it:

  The point $\mathbf{y}$ is an α-**limit point** of a point $\mathbf{x}$ if there is a sequence $t_n \to -\infty$ such that $\varphi_{t_n}(\mathbf{x}) \to \mathbf{y}$.

  The α-**limit set** $L_\alpha(\mathbf{x})$ is the set of α-limit points of $\mathbf{x}$.

  With $t_n \to +\infty$ instead, we obtain ω-**limit points**, and the ω-**limit set** $L_\omega(\mathbf{x})$.

  (As you might know, α and ω are the first and last letters of the Greek alphabet; cf. the following well-known passage from the Bible (Rev. 22:13): "I am the Alpha and the Omega, the first and the last, the beginning and the end." The terminology for limit sets is meant to convey the idea that $L_\alpha(\mathbf{x})$ and $L_\omega(\mathbf{x})$ give information about how the orbit of $\mathbf{x}$ behaves at the "beginning" of time $(t \to -\infty)$ and at the "end" of time $(t \to +\infty)$.)

- In three or more dimensions, limit sets can be extremely complicated, since trajectories have room to wind around in space in very strange ways. But in the plane, the possibilities are much more restricted, as shown by **Theorem 3.9.1**, the **Poincaré–Bendixson theorem** (given without proof in the textbook[1]):

  > If a compact nonempty limit set in the plane contains no equilibrium points, then it must be a periodic orbit.

  (There is also a more general version of the theorem, which says what can happen if the limit set contains finitely many equilibrium points; see Wikipedia: Poincaré–Bendixson theorem, for example.)

- Some properties of ω-limit sets:

  – $L_\omega(\mathbf{x})$ is always a **closed** and **invariant** set. (See **Definition 3.9.2**.)

[1] For proofs, see for example H. Amann, *Ordinary Differential Equations*, de Gruyter (1990), p. 333, or C. Chicone, *Ordinary Differential Equations with Applications*, Second Edition, Springer (2006), p. 101.

  - Limit sets may be empty, unbounded, disconnected (see problem A27). But if the forward orbit of $\mathbf{x}$ is **bounded**, then $L_\omega(\mathbf{x})$ is **connected**, **compact** and **non-empty**, and[2]

$$\varphi_t(\mathbf{x}) \to L_\omega(\mathbf{x}) \qquad \text{as } t \to \infty.$$

The corresponding properties hold of course for α-limit sets as $t \to -\infty$.

Proof of the invariance property. Suppose $\mathbf{y} \in L_\omega(\mathbf{x})$. By definition, this means that $\varphi_{t_n}(\mathbf{x}) \to \mathbf{y}$ for some sequence $t_n \to \infty$. Fix an arbitrary $t \in \mathbf{R}$. Applying the continuous function φ_t to both sides gives $\varphi_{t+t_n}(\mathbf{x}) \to \varphi_t(\mathbf{y})$, so $\varphi_t(\mathbf{y}) \in L_\omega(\mathbf{x})$. □

(I omit the proofs of the other properties, although they are not very difficult.)

- A **limit cycle** (**Definition 3.8.2**) is a periodic orbit which lies in the α- or ω-limit set of some point not on the orbit.

  To show the existence of a limit cycle for a planar system using the Poincaré–Bendixson theorem, one tries to find a **trapping region containing no equilibria**. A **trapping region** for a system with flow φ_t is a compact, connected set $D \subset \mathbf{R}^2$ such that $\varphi_t(D) \subset D$ for $t > 0$. When reading this definition, it's important to note that Arrowsmith & Place use the symbol "$\subset$" for *strict* set inclusion. The point is that if we only require D to be forward invariant (that is, $\varphi_t(D) \subseteq D$ for $t > 0$, with non-strict set inclusion "$\subseteq$"), then it may be the case that D is a union of periodic orbits[3], in which case there are no limit cycles in D. But with strict set inclusion, the points in the nonempty set $D \setminus \varphi_t(D)$ (for any fixed $t > 0$) can't lie on a periodic orbit[4], and the ω-limit set of such a point $\mathbf{x}_0$ is a nonempty compact subset of D. If we have chosen the trapping region D such that it contains no equilibria, the Poincaré–Bendixson theorem says that $L_\omega(\mathbf{x}_0)$ must be a periodic orbit. Since $\mathbf{x}_0$ was not on any periodic orbit, $L_\omega(\mathbf{x}_0)$ doesn't contain $\mathbf{x}_0$, so it is a periodic orbit which is the ω-limit set of a point not on the orbit – in other words, it's an ω-limit cycle. Conclusion: the trapping region D contains *at least one* limit cycle.

  (But there may be many limit cycles in D! To prove that there is *at most one* limit cycle is usually much more difficult.)

- Now that we know what an ω-limit set $L_\omega(\mathbf{x})$ is, we can state **LaSalle's invariance principle** that was mentioned in the previous lecture. We consider a system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ with flow φ_t.

  Theorem. Suppose $V : \Omega \to \mathbf{R}$ is differentiable on the open set $\Omega \subseteq \mathbf{R}^n$, and satisfies $\dot{V}(\mathbf{y}) \le 0$ for each $\mathbf{y}$ in some closed set $M \subseteq \Omega$.

  1. If $\mathbf{x}$ is a point in M whose forward orbit $\mathscr{O}^+(\mathbf{x})$ never leaves M, then there is an $\alpha \in \mathbf{R}$ such that

$$L_\omega(\mathbf{x}) \subseteq \{\mathbf{y} \in M : V(\mathbf{y}) = \alpha\}.$$

     This implies that $L_\omega(\mathbf{x})$ is an invariant set contained in the set

$$C = \{\mathbf{y} \in M : \dot{V}(\mathbf{y}) = 0\},$$

     and hence $L_\omega(\mathbf{x}) \subseteq E$, where E is the *largest invariant subset* of C (i.e., E is the union of all trajectories which stay in C for all $t \in \mathbf{R}$).

     Proof. This is trivially true if $L_\omega(\mathbf{x}) = \emptyset$, since the empty set is a subset of *every* set. So assume $L_\omega(\mathbf{x}) \ne \emptyset$. (In particular, this assumption entails that the solution $\varphi_t(\mathbf{x})$ exists for all $t \ge 0$.)

[2] The notation $\varphi_t(\mathbf{x}) \to L_\omega(\mathbf{x})$ means that for any open set $U \supset L_\omega(\mathbf{x})$ there is a time T such that $\varphi_t(\mathbf{x}) \in U$ for all $t > T$.

[3] For example (in polar coordinates), if the system is $\dot{r} = 0$, $\dot{\theta} = 1$, and D is the annulus $1 \le r \le 2$.

[4] Suppose that $\mathbf{x}_0 \in D \setminus \varphi_t(D)$ for some $t > 0$ and that $\mathbf{x}_0$ lies on a periodic orbit with period $T > 0$. Let n be a positive integer such that $nT > t$. Then

$$\mathbf{y} = \varphi_{nT-t}(\mathbf{x}_0) \in \varphi_{nT-t}(D) \subset D$$

so that

$$\mathbf{x}_0 = \varphi_{nT}(\mathbf{x}_0) = \varphi_t(\mathbf{y}) \in \varphi_t(D),$$

which is a contradiction.

To begin with,

$$\begin{aligned} L_\omega(\mathbf{x}) &\subseteq \overline{\mathscr{O}^+(\mathbf{x})} && \text{(follows from def. of } L_\omega(\mathbf{x})\text{)} \\ &\subseteq \overline{M} && \text{(since } \mathscr{O}^+(\mathbf{x}) \subseteq M \text{ by assumption)} \\ &= M && \text{(since } M \text{ is closed).} \end{aligned}$$

Next, the assumptions that $\dot{V} \le 0$ on M and that $\varphi_t(\mathbf{x})$ stays in M for $t \ge 0$ imply that $V(\varphi_t(\mathbf{x}))$ is a weakly decreasing function of t for $t \ge 0$, so the limit

$$\alpha = \lim_{t\to\infty} V(\varphi_t(\mathbf{x}))$$

exists, either as a real number $\alpha \in \mathbf{R}$ or in the improper sense $\alpha = -\infty$. But if $\mathbf{y}$ is any element in the nonempty set $L_\omega(\mathbf{x})$, meaning that $\varphi_{t_n}(\mathbf{x}) \to \mathbf{y}$ for some sequence $t_n \nearrow \infty$, then

$$V(\mathbf{y}) = V\Big(\lim_{n\to\infty} \varphi_{t_n}(\mathbf{x})\Big) = \lim_{n\to\infty} V\big(\varphi_{t_n}(\mathbf{x})\big) = \alpha,$$

since V is continuous. This shows that α equals the real number $V(\mathbf{y})$, not $-\infty$.

The above calculation holds for an arbitrary $\mathbf{y} \in L_\omega(\mathbf{x})$, so $V = \alpha$ on all of $L_\omega(\mathbf{x})$. And $L_\omega(\mathbf{x})$ is an invariant set (general property of limit sets), so $\varphi_t(\mathbf{y}) \in L_\omega(\mathbf{x})$ for all t if $\mathbf{y} \in L_\omega(\mathbf{x})$. Thus $V(\varphi_t(\mathbf{y})) = \alpha$ for all t, and hence

$$\dot{V}(\mathbf{y}) = 0 \quad \text{if } \mathbf{y} \in L_\omega(\mathbf{x}).$$

What we have shown now is that $L_\omega(\mathbf{x})$ is an invariant set which is contained in the set $C \subseteq M$ where $\dot{V} = 0$. Therefore, it must trivially be contained in E, the *largest* invariant set contained in C. □

2. If moreover the forward orbit $\mathscr{O}^+(\mathbf{x})$ is **bounded**, then $L_\omega(\mathbf{x})$ is nonempty and $\varphi_t(\mathbf{x}) \to L_\omega(\mathbf{x})$ as $t \to \infty$. So $\varphi_t(\mathbf{x}) \to E$ as $t \to \infty$.

   Proof. The first sentence was one of the general properties of limit sets stated at the beginning of the lecture. The second sentence follows at once from the property $L_\omega(\mathbf{x}) \subseteq E$ that we proved in item 1. (The conclusion that $\varphi_t(\mathbf{x}) \to E$ is of course a bit weaker than $\varphi_t(\mathbf{x}) \to L_\omega(\mathbf{x})$, but the point is that the set E does not depend on $\mathbf{x}$.) □

3. If M is **compact** and **forward invariant**, then items 1 and 2 apply to *every* point $\mathbf{x} \in M$. So in this case, $\varphi_t(\mathbf{x}) \to E$ as $t \to \infty$, for every $\mathbf{x} \in M$.

   Proof. Trivial. □

The point of this theorem is that the set E is often quite easy to determine. We find the set C simply by computing $\dot{V}$ and checking where it's zero. This is typically some curve, if we are in $\mathbf{R}^2$. Then we study what the vector field $\mathbf{X}$ is doing at each point of the set C – if the vector field is pointing out from C at some point, then that point can't be part of a trajectory completely contained in C, so it can't belong to E.

A typical application is the situation described in **Theorem 3.5.2** (see the previous lecture), where we have only managed to find a *weak* Liapunov function V, but the "bad" set C where we have $\dot{V} = 0$ instead of $\dot{V} < 0$ doesn't contain any trajectories except the equilibrium point $\mathbf{x}^*$.

Proof of Theorem 3.5.2. As usual, let $B = \overline{B(\mathbf{x}^*, r)}$ be a closed ball (or some other neighbourhood of $\mathbf{x}^*$ topologically equivalent to a closed ball) contained in Ω, and define

$$N = \big\{\mathbf{x} \in B : V(\mathbf{x}) < \alpha\big\}, \quad \text{where } \alpha = \min_{\mathbf{x}\in\partial B} V(\mathbf{x}) > 0.$$

Stability of $\mathbf{x}^*$ follows from the weak version of Liapunov's theorem, so we just need to show that N is a domain of stability. To apply LaSalle's invariance principle, we need a compact and forward invariant set M, so N (which is open) won't do. Instead, take β with $0 \le \beta < \alpha$, and let

$$M = \big\{\mathbf{x} \in B : V(\mathbf{x}) \le \beta\big\}.$$

Then M is closed (and hence compact) since V is continuous.[5] And it's forward invariant; indeed, a trajectory starting in M can't leave M, since then it would enter the part of B where $V > \beta$, so V wouldn't be weakly decreasing along that trajectory, and that would contradict the assumption that $\dot{V} \le 0$ in Ω. Now the invariance principle says that every trajectory starting in M converges to E, the largest invariant subset of $C = \{\mathbf{x} \in M : \dot{V}(\mathbf{x}) = 0\}$. But by assumption, there are no trajectories even in the larger set $C_2 = \{\mathbf{x} \in \Omega : \dot{V}(\mathbf{x}) = 0\}$ except for the equilibrium $\mathbf{x}^*$. Hence $E = \{\mathbf{x}^*\}$, and every trajectory starting in M converges to $\mathbf{x}^*$. So every such set M is a domain of stability.

To show that N is a domain of stability, just note that any $\mathbf{x}_0 \in N$ belongs to the set $M \subseteq N$ defined using $\beta = V(\mathbf{x}_0) < \alpha$. Therefore the trajectory starting at $\mathbf{x}_0$ stays in N and converges to $\mathbf{x}^*$. □

- The **Poincaré map** (or **first-return map**) associated with a periodic orbit is defined on p. 104.

- **Theorem 3.9.2** is called the **Bendixson criterion**.

  A simple generalization (with virtually the same proof) is the **Bendixson–Dulac criterion**, which gives the same conclusion provided that that there is a function $f(x_1, x_2)$ of class C^1 such that the divergence of the rescaled vector field $f\mathbf{X}$,

$$\nabla \cdot (f\mathbf{X}) = \frac{\partial}{\partial x_1}(fX_1) + \frac{\partial}{\partial x_2}(fX_2),$$

  is of constant sign (positive or negative) in D.

  (One can in fact weaken the hypotheses, by allowing $\nabla \cdot (f\mathbf{X})$ to be zero on a set of measure zero; this does not alter the fact that the double integral in the proof must be nonzero.)

Exercises

- α- and ω-limit sets: 3.35, A26, A27*.

  (There's an error in the answer to 3.35b; it should say $L_\alpha(\mathbf{x}) = \{\mathbf{0}\}$ for $0 < r < 1$.)

- Poincaré map: 3.36. (In this problem, it's understood that $a > 0$.)

- Invariant sets, trapping regions: 3.42, 3.43, A28.

  - Don't miss the follow-up question that's formulated at the end of problem 3.42, after part (e).
  - **Problem 3.42a is incorrect**; the upper half-plane $x_2 \ge 0$ is actually **not** a positively invariant set for that system! The reason is rather subtle; can you see what it is that goes wrong compared to Definition 3.9.2?
  - In problem 3.42d, it's fine to "cheat" a little by using the computer to draw the level curves of $3(x_1^2 + x_2^2) - 2x_1^3$.
  - In the last part of problem 3.43 ("Show that the system has a limit cycle when $F = 0$") it is assumed that $w \ne 0$.

- The Bendixson criterion: A29.

[5] If $\mathbf{x}_n$ is a sequence of points in M converging to $\mathbf{x}$, then $\mathbf{x} \in \overline{B} = B$, and by continuity

$$\underbrace{V(\mathbf{x}_n)}_{\le \beta} \to V(\mathbf{x}),$$

so $V(\mathbf{x}) \le \beta$. Hence $\mathbf{x} \in M$, which means that M is closed.

Additional problems

A26 Show that for the equation $\dot{x} = 1$, the ω-limit set of each point is the **empty** set.

A27 Sketch the phase portrait for the system

$$\dot{x} = -y + \frac{x}{1+x^2}, \qquad \dot{y} = x(1-y^2).$$

Show that the ω-limit set of any point $(x,y) \neq (0,0)$ in the strip $|y| < 1$ is the union of the lines $y = \pm 1$, and hence is **unbounded** and **disconnected**. (Note also that for these points it's *not* true that $\varphi_t(x,y) \to L_\omega(x,y)$ as $t \to \infty$.) Hint.

A28 Show that the parabola $y = x^2$ is an invariant set for the system

$$\dot{x} = x^2 - x - y, \qquad \dot{y} = x^2 - 3y.$$

(Don't forget to show that the solutions starting on the parabola *exist* for all $t \in \mathbf{R}$, since this is part of the definition of "invariant set".) Sketch the phase portrait. Hints.

A29 Show that the following systems have no closed orbits:

(a) $\dot{x} = y + x^3$, $\dot{y} = x + y + y^3$. (Use Bendixson.)

(b) $\dot{x} = y$, $\dot{y} = -x - y + x^2 + y^2$. (Use Bendixson–Dulac with $f(x,y) = e^{-2x}$.)

Lecture 8. Some applications

(Arrowsmith & Place, sections 5.1, 5.2, 5.3, 5.4.)

- In class we will look at a selection of the applications from Chapter 5, but there will not be time to cover everything, so you'll have to read the rest for yourself.
- The analysis in Section 5.3.3 of the **Holling–Tanner predator–prey model**

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{wxy}{D+x}, \qquad \frac{dy}{dt} = sy\left(1 - \frac{y}{x/J}\right)$$

can be made simpler by writing the system in **dimensionless variables** (τ, u, v) instead of (t, x, y). This **nondimensionalization** is a very useful technique for reducing the number of parameters in a system, and since it's not described in the textbook, I'll explain it here instead. Let

$$t = c_0 \tau, \qquad x = c_1 u, \qquad y = c_2 v,$$

where c_0, c_1 and c_2 are constants that we will specify soon. Inserting this into the differential equations, we get

$$\frac{c_1}{c_0}\frac{du}{d\tau} = rc_1 u\left(1 - \frac{c_1 u}{K}\right) - \frac{wc_1 u c_2 v}{D + c_1 u}, \qquad \frac{c_2}{c_0}\frac{dv}{d\tau} = sc_2 v\left(1 - \frac{c_2 v}{c_1 u/J}\right),$$

which we can simplify to

$$\frac{du}{d\tau} = \underset{\uparrow}{(rc_0)}\, u\left(1 - \underset{\uparrow}{\frac{c_1}{K}}\, u\right) - \frac{\frac{wc_0c_2}{c_1} uv}{\frac{D}{c_1} + u}, \qquad \frac{dv}{d\tau} = (sc_0)\, v\left(1 - \underset{\uparrow}{\frac{c_2 J}{c_1}}\, \frac{v}{u}\right).$$

At this stage, we have some freedom of choice, but for example we can get rid of the coefficients indicated by the arrows, if we choose c_0, c_1 and c_2 such that

$$rc_0 = 1, \qquad \frac{c_1}{K} = 1, \qquad \frac{c_2 J}{c_1} = 1.$$

In other words, we take

$$c_0 = \frac{1}{r}, \qquad c_1 = K, \qquad c_2 = \frac{K}{J}. \tag{1}$$

Then the equations become

$$\frac{du}{d\tau} = u(1-u) - \frac{\frac{w}{rJ}uv}{\frac{D}{K}+u}, \qquad \frac{dv}{d\tau} = \tfrac{s}{r}\, v\left(1-\frac{v}{u}\right),$$

and if we now give names to the remaining coefficients appearing in the formulas, for example

$$\alpha = \frac{w}{rJ}, \qquad \beta = \frac{s}{r}, \qquad \delta = \frac{D}{K}, \tag{2}$$

then the system in its final form is

$$\frac{du}{d\tau} = u(1-u) - \frac{\alpha uv}{\delta+u}, \qquad \frac{dv}{d\tau} = \beta v\left(1-\frac{v}{u}\right). \tag{3}$$

Note that this system contains only three parameters (α,β,δ), instead of the original six parameters (r,K,w,D,s,J). By using the possibility of rescaling the three variables, we have reduced the number of parameters by three.

Our choice (1) of the constants c_k means that the new variables the we have introduced are actually

$$\tau = \frac{t}{c_0} = rt, \qquad u = \frac{x}{c_1} = \frac{x}{K}, \qquad v = \frac{y}{c_2} = \frac{Jy}{K}.$$

Considering that the parameter r in the original ODEs must have the dimension $[\text{time}]^{-1}$ (it's a per capita growth rate), and that t of course has dimension [time], we see that the rescaled time variable $\tau = rt$ is actually dimensionless. Similarly, the carrying capacity K for the prey has the same dimension as the prey population size x (whatever unit we happen to use for this, like number of millions of individuals, or biomass in kilograms, or something else), so the variable $u = x/K$ is dimensionless. And so is v, as you can check.

Moreover, the new parameters given by (2) are also dimensionless. For example, r and s are both per capita growth rates and have the same units, so $\beta = s/r$ is a dimensionless quantity which measures the *ratio* between the intrinsic growth rates of the two species. It is rather meaningless to say something like "r is small", since this depends on what time unit we are using – if we switched from measuring time in nanoseconds to measuring it in centuries, we would get a very different numerical value for r. But the statement "β is small" (say much less than 1) expresses a fact which is meaningful regardless of scale, namely that the predators reproduce much slower than the prey.

With the system in the simpler form (3), we can now carry out the same analysis as in Section 5.3.3, but it will be cleaner, since there is less to write, and we also don't get all the original parameters scattered among our formulas, but we always keep them gathered in the meaningful combinations α, β and δ. In the textbook, they do a bit of rescaling at the end, namely taking x/x^* and y/y^* as new variables, where (x^*, y^*) is the nontrivial equilibrium point, but they don't use the possibility of rescaling time to get rid of one more parameter.

- The very last sentence in Section 5.3 (on p. 188) is (with my emphasis) "Thus the phase portrait corresponding to Fig. 5.22(a) **has no limit cycle**; (y_1^*, y_2^*) is simply a stable focus." However, the claim that there cannot exist any limit cycles in this case is **false**; there are actually parameter values such that the stable focus is surrounded by two limit cycles, the inner one unstable and the outer one stable.[1]

[1] A. Gasull, R. E. Kooij & J. Torregrosa, Limit cycles in the Holling–Tanner model, Publicacions Matemàtiques, Vol. 41 (1997), 149–167.

[Table of contents]

Exercises

- Damped harmonic oscillator: 5.2, 5.3, 5.10.
- Population models: 5.15.

  Since populations can't be negative, it's enough here to draw the phase portrait in the closed positive quadrant ($x_1 \geq 0$ and $x_2 \geq 0$).

  Please note that the book's answer is not completely correct; the origin is an unstable *star* node, where *every* direction is principal, not just the directions $(1,0)$ and $(0,1)$! Also, when sketching the phase portrait, take particular care to draw the improper nodes at $(2,0)$ and $(0,2)$ correctly – it's easy to get them wrong!

  A bonus question: What would change if the coefficents were less "symmetric"? For example, consider instead $\dot{x}_1 = x_1(2 - x_1 - 2x_2)$, $\dot{x}_2 = x_2(3 - 3x_1 - x_2)$. For your convenience, here are Wolfram Alpha links if you want to check your answers: symmmetric case, non-symmetric case.
- Epidemics: A30, A31.
- The chemostat: A32, A33, A34, A35, A36, A37, A38, A39.

Additional problems

A30 In the basic **SIR model** for the spread of an infectious disease in a population, the symbols S, I and R denote the fractions of the total population that are **susceptible** to infection, **infected**, and **recovered** (permanently immune to the infection). This means that $S, I, R \in [0,1]$, and $S + I + R = 1$. Births and deaths are ignored, so that the population is assumed to be constant during the time interval considered.

The ODEs describing the time evolution of $S(t)$, $I(t)$ and $R(t)$ in this model are

$$dS/dt = -\alpha IS, \qquad dI/dt = \alpha IS - \beta I, \qquad dR/dt = \beta I,$$

where the parameters α and β are positive. The first idea here is that people get infected at a rate proportional to the product SI, which is a measure of how often a susceptible individual encounters an infected one. The rate of decrease of S must be the same as the rate of increase of I, since individuals just "move from the S group to the I group"; hence the terms $\pm\alpha IS$ is the ODES. And the second idea is that people recover, i.e., "move from the I group to the R group", at a constant rate, so that the number of recoveries during a short time interval is simply proportional to the number of infected individuals at that instant of time; hence the terms $\pm\beta I$ in the ODEs.

(a) Check that the quantity $\frac{d}{dt}(S + I + R) = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$ is equal to zero.
(This means that solutions to the ODEs have the property that $S(t) + I(t) + R(t)$ is constant in time, which is good since $S + I + R = 1$ is supposed to hold always. To be explicit: if we solve the ODEs with initial values satisfying $S(0) + I(0) + R(0) = 1$, then we know that the solution also satisfies $S(t) + I(t) + R(t) = 1$ for all t, as it should.)

(b) The relation $R = 1 - I - S$ implies that the quantity R is redundant, so that it's enough to study the two-dimensional dynamical system

$$dS/dt = -\alpha IS, \qquad dI/dt = \alpha IS - \beta I$$

for $S(t)$ and $I(t)$. Our goal here is to construct the phase portrait for this system. Clearly, since $S, I \in [0,1]$, it's enough to consider the unit square in the SI-plane. But actually it's enough to consider just *half* that square, namely the triangular region

$$D = \{(S, I) \in \mathbf{R}^2 : S \geq 0,\ I \geq 0,\ S + I \leq 1\}.$$

Explain why!

(c) Draw the nullclines $\dot{S} = 0$ and $\dot{I} = 0$ in the SI-plane, and mark the regions that are formed with RU/RD/LU/LD, as usual. (We will restrict our attention to the triangle D later, but for now you might as well draw the picture in the whole SI-plane.)

What are the equilibrium points?

(d) Express the flow lines on the form $I = I(S)$ by eliminating t in the usual way, and solving the resulting ODE:

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \cdots$$

Draw the family of curves $I = I(S)$ as precisely as you can!

(e) Add directions to your curves, in accordance with the directions (UR, etc.) that you determined above. And look at the part of your picture which lies in the triangle D, to get an idea of how the epidemic evolves.

There will be two cases, $0 < \beta/\alpha < 1$ and $1 < \beta/\alpha$, which will look different since in the first case the nullcline $S = \beta/\alpha$ passed through the region D, while in the second case it doesn't. So make two separate drawings of the phase portrait in the region D, one for each case.

(As is commonly done, we ignore the borderline case $\beta/\alpha = 1$ here, since "the probability that it will occur in reality is zero", and the model is just an approximation anyway.)

(f) How does the epidemic evolve? In particular, what happens as $t \to \infty$? Can you interpret the two cases, considering the meaning of the parameters α and β? Solutions.

A31 Let S, I and R be as in problem A30 above. If we assume that immunity is only temporary, so that recovered individuals may go back to the susceptible class after a while, we get an **SIRS model**:

$$dS/dt = -\alpha IS + \gamma R, \qquad dI/dt = \alpha IS - \beta I, \qquad dR/dt = \beta I - \gamma R,$$

where α, β and γ are all positive. (The parameters α and β are as before, while γ measures the rate at which immunity is lost.)

Again, $\frac{d}{dt}(S + I + R) = 0$, so that the ODEs are consistent with the requirement that $S + I + R = 1$. And therefore, we can again use $R = 1 - S - I$ to eliminate R and get the following two-dimensional dynamical system for $S(t)$ and $I(t)$:

$$dS/dt = -\alpha IS + \gamma(1 - S - I), \qquad dI/dt = \alpha IS - \beta I.$$

And again, our goal is to construct the phase portrait for this system in the triangular region

$$D = \{(S, I) \in \mathbf{R}^2 : S \geq 0,\ I \geq 0,\ S + I \leq 1\}.$$

(a) Show that the S-nullcline for the two-dimensional system can be rewritten as follows:

$$-\alpha IS + \gamma(1 - S - I) = 0 \quad \Longleftrightarrow \quad \left(S + \frac{\gamma}{\alpha}\right)\left(I + \frac{\gamma}{\alpha}\right) = \frac{\gamma}{\alpha}\left(1 + \frac{\gamma}{\alpha}\right).$$

Use this to draw the curve in the SI-plane! Draw the whole curve just for practice – we will restrict our attention to the triangle D later. Part of the curve will in fact go inside the triangle D, but you will need to give an argument to motivate *why* that's the case!

(Hint: You should know very well already what a curve of the form "SI = constant" looks like. From this you should be able to figure out what a curve of the form "$(S - a)(I - b)$ = constant" looks like!)

Remark: It's also possible to draw the curve by rewriting it as

$$-\alpha IS + \gamma(1 - S - I) = 0 \quad \Longleftrightarrow \quad I = I(S) = \frac{1 - S}{1 + \frac{\alpha}{\gamma} S},$$

and then using calculus to investigate the function $I(S)$. But this would involve a bit more work, such as computing limits to determine asymptotes, and investigating the signs of the first derivative $I'(S)$ and the second derivative $I''(S)$. (You really need to consider convexity to make sure that the curve goes inside D.)

(b) Use the Bendixson–Dulac criterion, with the scaling factor $1/I$, to show that there cannot be any limit cycles in the interior of the triangular region D. That is, compute the divergence of the vector field

$$\frac{1}{I}\left(\frac{dS}{dt},\frac{dI}{dt}\right)=\left(\frac{dS/dt}{I},\frac{dI/dt}{I}\right)=\left(\frac{-\alpha IS+\gamma(1-S-I)}{I},\frac{\alpha IS-\beta I}{I}\right)$$

and study its sign. (This will be useful when drawing the phase portrait below, for arguing that all solutions must converge to an equilibrium, since they can't go into a cycle.)

(c) Consider the case $0<\alpha<\beta$. Draw the nullclines (for both S and I together) in the SI-plane. Are there any equilibrium points in the triangle D? If so, analyze them using linearization. Sketch the phase portrait in the region D, and give a biological interpretation of the results.

(d) Do the same for the case $0<\beta<\alpha$.

A32 A **chemostat** is a bioreactor for growing microorganisms in the laboratory. The liquid in the reactor tank is kept well stirred at all times, and contains the microbial culture as well as nutrients needed for the microorganisms to grow. All nutrients are supplied in excess, except for one, called the *limiting nutrient* (since it's the available amount of this nutrient that will limit how fast the microorganism population can grow). Let $C=C(t)$ denote the concentration of the limiting nutrient in the tank, as a function of time t, and let $X=X(t)$ be the concentration of microorganisms; both are measured in units of mass per volume. The volume V of liquid in the tank is kept constant by continuously harvesting microorganism–nutrient solution at a constant rate F (units: volume per time), and resupplying fresh nutrient solution at the same rate F; the ratio $D=F/V$ is called the *dilution rate* (unit: 1/time). The growth rate of the microorganisms can be controlled by adjusting the concentration C_{in} of the limiting nutrient in the solution which is being pumped into the reactor. See for example the Wikipedia article about the chemostat for schematic diagrams and more information.

We are going to investigate the following mathematical model of a chemostat:

$$\underbrace{\frac{dX}{dt}}_{\substack{\text{organism}\\\text{rate of}\\\text{change}}}=\underbrace{f(C)X}_{\substack{\text{organism}\\\text{growth}\\\text{rate}}}-\underbrace{DX}_{\substack{\text{organism}\\\text{outflux}\\\text{rate}}},\qquad \underbrace{\frac{dC}{dt}}_{\substack{\text{nutrient}\\\text{rate of}\\\text{change}}}=\underbrace{DC_{\text{in}}}_{\substack{\text{nutrient}\\\text{influx}\\\text{rate}}}-\underbrace{DC}_{\substack{\text{nutrient}\\\text{outflux}\\\text{rate}}}-\underbrace{f(C)X/\gamma}_{\substack{\text{nutrient}\\\text{consumption}\\\text{rate}}},$$

where the per-capita growth rate of the microorganisms is described by the function

$$f(C)=\frac{K_{\max}C}{K_m+C}.$$

The dimensionless constant γ is called the *yield*, since it determines how many mass units of microorganisms that are obtained per mass unit of nutrient consumed. The formula for the function $f(C)$ is an empirical expression suggested by the famous French biochemist Jacques Monod. One of the parameters in this expression, $K_{\max}$, is the greatest possible growth rate of the microorganisms, obtained when there is an infinite supply of the limiting nutrient:

$$\lim_{C\to\infty}f(C)=\lim_{C\to\infty}\frac{K_{\max}C}{K_m+C}=\lim_{C\to\infty}\frac{K_{\max}}{K_m\cdot\frac{1}{C}+1}=\frac{K_{\max}}{K_m\cdot 0+1}=K_{\max}.$$

The other parameter, K_m, is the value of C for which the growth rate is half the maximal rate:

$$f(K_m)=\left.\frac{K_{\max}C}{K_m+C}\right|_{C=K_m}=\frac{K_{\max}K_m}{K_m+K_m}=\frac{K_{\max}}{2}.$$

So in total there are five parameters in the model:

- The constants $K_{\max}$, K_m and γ, which describe properties of the microorganism in question.

- The values C_{in} and D, which you can adjust in the settings of the apparatus. (The parameters F and V don't appear individually in the model, only in the combination $D = F/V$.)

(a) Before we start analyzing the ODEs mathematically, let's do some thought experiments!

Suppose that you are working in a lab where you are growing microorganisms in a chemostat. Everything is functioning as it should: the microorganism–nutrient solution in the reactor tank is at an equilibrium state, meaning that the concentrations $X(t)$ and $C(t)$ stay constant at some levels X^* and C^*, so that you are harvesting microorganisms at a constant rate.

One day your boss tells you that you will need to increase the production. You get the obvious idea that the microorganisms will grow faster if they get more food, so in the evening you turn the appropriate knob on the apparatus to increase the concentration C_{in} of the limiting nutrient which is fed into the chemostat. When you return the next morning, the machine has settled down into an equilibrium state again. The new equilibrium value X^* of the microorganism concentration is indeed higher than before, as you planned, but you are slightly surprised to find that the equilibrium value C^* of the limiting nutrient in the tank is exactly the same as before, even though you're feeding more nutrient into the tank. Can you think of an explanation for this fact?

(b) A while later, your boss says that the lab needs to save some money, so you will have to cut down on the amount of nutrient that you're giving the microorganisms. So you turn the same knob, but this time to decrease the value of C_{in}. When equilibrium is reached again, C^* is unchanged, which now doesn't surprise you anymore, since you have done part (a) of this problem! You decrease C_{in} even further, and C^* still doesn't change. You decrease C_{in} again, this time to a value that's lower than C^*. Then, when you return the next morning, you are shocked to find that there are no microorganisms left in the tank (i.e., $X^* = 0$) and that C^* has decreased to C_{in}. What happened?

A33 (a) Now let's start analyzing the chemostat model!

Your first task is to write down the ODEs

$$\frac{dx}{d\tau} = \cdots, \qquad \frac{dc}{d\tau} = \cdots$$

which are obtained when making the change of variables

$$t = a_0\tau, \qquad X = a_1 x, \qquad C = a_2 c$$

in the ODEs for the chemostat.

(b) Next, choose the constants a_0, a_1 and a_2 in such a way that the system you obtained in part (a) takes the form

$$\frac{dx}{d\tau} = \left(\frac{\beta_1 c}{\beta_2 + c} - 1\right) x, \qquad \frac{dc}{d\tau} = 1 - c - \frac{\beta_1 c x}{\beta_2 + c}.$$

Please state clearly in your answer how the scaling factors a_0, a_1, a_2 and the new parameters β_1, β_2 are defined in terms of the parameters in the original equations ($K_{\max}$, K_m, γ, C_{in} and D).

(c) The ODEs obtained in part (b) are supposed to be a *nondimensionalized* version of the chemostat model, so please perform the following checks:

- Verify that the new parameters β_1 and β_2, as you definied them in your answer to part (b), are really dimensionless.
- Similarly, verify that your a_0, a_1 and a_2 have the dimensions that they should have. (Namely, the same as t, X and C, respectively, so that the new variables $\tau = t/a_0$, $x = X/a_1$ and $c = C/a_2$ become dimensionless too.)

A34 Our goal is now to construct the phase portrait for the nondimensionalized system from problem A33, in the nonnegative quadrant of the xc-plane. Let's start with the equilibrium points.

(a) Find all equilibrium points for the nondimensionalized system.

(Spoilers: There is one rather trivial equilibrium point, the "washout" equilibrium, which is independent of the values of β_1 and β_2. Provided that $\beta_1 \neq 1$ and $\beta_1 \neq 1+\beta_2$, there is also another equilibrium point which is more interesting; call it (x^*, c^*).)

(b) Show that (x^*, c^*) always lies on the line $x + c = 1$.

(c) Show that (x^*, c^*) lies in the positive quadrant (i.e., $x^* > 0$ and $c^* > 0$) if and only if β_1 belongs to the open interval $(1+\beta_2, \infty)$. We will call this the "good" case.

(d) We will also consider two "bad" cases, with β_1 in the interval $(0,1)$ or $(1, 1+\beta_2)$. In which quadrant does (x^*, c^*) lie in those two cases?

(As usual, we ignore the borderline cases $\beta_1 = 1$ and $\beta_1 = 1 + \beta_2$.)

(e) Compute the system's Jacobian matrix $J(x,c)$, and in particular $J(0,1)$ and $J(x^*, c^*)$.

Solutions.

A35 (a) Determine the eigenvalues and eigenvectors of $J(0,1)$ from problem A34e.

(b) Use the eigenvalues to determine the stability and type (focus/node/saddle, etc.) of the washout equilibrium $(0,1)$ in all three cases:

- the first bad case $\beta_1 \in (0,1)$,
- the second bad case $\beta_1 \in (1, 1+\beta_2)$,
- the good case $\beta_1 \in (1+\beta_2, \infty)$.

(Do *not* use the trace–determinant criterion! It's much easier to determine the type directly from the eigenvalues here.)

(c) Determine from $J(x^*, c^*)$ whether the nontrivial equilibrium (x^*, c^*) is stable or unstable in the good case $\beta_1 \in (1+\beta_2, \infty)$.

(*Here* it's very convenient to use the trace–determinant criterion.)

Remark 1: Although $J(x^*, c^*)$ may look a little "nasty", it's actually not that difficult to find its eigenvalues and eigenvectors, so you can try that if you like. This will quickly give you information about what type of equilibrium it is, not just whether it's stable or not. But we will soon obtain this information in another way (problem A36c), so it's not really necessary to do it at this stage.

Remark 2: The bad cases are not relevant for (x^*, c^*), since we are only interested in the phase portrait in the nonnegative quadrant, and we have seen that (x^*, c^*) only lies there in the good case.

A36 (a) Show, from the ODEs, that the point $\big(x(\tau), c(\tau)\big)$ moves in such a way that its perpendicular distance to the line $x + c = 1$ decreases monotonically towards zero (or is identically zero).

Hint.

(b) Explain why this implies that no equilibrium can be a focus!

(c) Together with what we found in problem A35c, what does this tell us about the type of the equilibrium (x^*, c^*) in the good case?

A37 Now it's time to look at the nullclines. Recall that our system is

$$x' = \left(\frac{\beta_1 c}{\beta_2 + c} - 1\right) x, \qquad c' = 1 - c - \frac{\beta_1 c x}{\beta_2 + c}.$$

The x-nullcline is clearly just the union of two straight lines:

$$\left(\frac{\beta_1 c}{\beta_2 + c} - 1\right) x = 0 \qquad \Longleftrightarrow \qquad x = 0 \quad \text{or} \quad c = c^* = \frac{\beta_2}{\beta_1 - 1}.$$

The c-nullcline is a curve, which can be expressed with x as a function of c:

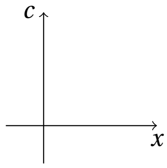
$$1 - c - \frac{\beta_1 c x}{\beta_2 + c} = 0 \qquad \Longleftrightarrow \qquad x = \frac{(1-c)(\beta_2 + c)}{\beta_1 c} = -\frac{1}{\beta_1} c + \frac{1-\beta_2}{\beta_1} + \frac{\beta_2}{\beta_1 c} =: g(c).$$

(a) Use calculus to draw this curve $x = g(c)$ in the xc-plane, in the good case $\beta_1 \in (1 + \beta_2, \infty)$. (In the end we will only be interested in nonnegative x and c, but you should draw the curve in the whole xc-plane, just for practice!) Also draw the line $x + c = 1$ in the same figure.

Solution.

(b) Do the same for the two bad cases.

A38 Now we can finally draw the phase portrait for the nondimensionalized chemostat model! To facilitate the marking of the homework, please draw the x-axis horizontally and the c-axis vertically, like this:



(a) Sketch the phase portrait in the nonnegative quadrant of the xc-plane, in the good case $\beta_1 \in (1 + \beta_2, \infty)$.

(As usual, consider the nullclines and the signs of $dx/d\tau$ and $dc/d\tau$. But you should also try to use all the other information that you have gathered above, such as the type of the equilibrium points, the principal directions, the important observations about the line $x + c = 1$, etc.)

(b) The same, but for the first bad case $\beta_1 \in (0, 1)$.

(c) And the same again, but for the second bad case $\beta_1 \in (1, 1 + \beta_2)$.

A39 It only remains to translate these results back to the original chemostat model, and think a little about whether they agree with our intuition, and in particular with the thought experiments from problem A32.

(a) By now it should hopefully be clear what is "good" and "bad" about the different cases. Show that the condition $\beta_1 > 1$ corresponds to $K_{\text{max}} > D$ in terms of the original parameters, and explain why this condition should obviously be necessary for avoiding washout.

(b) But $\beta_1 > 1$ is not sufficient to avoid washout; we need the stronger condition $\beta_1 > 1 + \beta_2$. In order to understand this condition, express it in terms of the original parameters, and show that (provided the condition $K_{\text{max}} > D$ holds to begin with) it can be rewritten as $C_{\text{in}} > C^*$, where

$$C^* = \frac{K_m D}{K_{\text{max}} - D}$$

is the equilibrium value of C that corresponds to $c = c^*$.

Why does it make sense that C_{in} ought to be greater than C^* in order to avoid washout? What's the sign of dC/dt if $C_{\text{in}} \le C^*$?

(c) Note that the expression above for the equilibrium nutrient concentration C^* doesn't depend on the nutrient supply concentration C_{in}.

Give a conceptual explanation of why it must be like that, as follows: Why should the equation $f(C^*) = D$ hold at a nontrivial equilibrium? (This can be seen from the ODE for X. But also think about what it *means* biologically.) Why does this condition determine C^* uniquely (if $K_{\text{max}} > D$)?

[Table of contents]

Lesson 3

Lecture 9. More about existence and uniqueness

(Not covered in Arrowsmith & Place; see notes below instead.)

Our goal this time is to use **Picard iteration**, also known as the **method of successive approximations**, to prove the **Picard–Lindelöf theorem**, the fundamental existence and uniqueness theorem for a system of (non-autonomous) first order ODEs $\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x})$ with a given initial condition $\mathbf{x}(t_0) = \mathbf{c}$.

Preliminaries from analysis: uniform convergence

We will need some theorems about convergence of sequences and series of *functions* (not just *numbers*).

Definition (Pointwise and uniform convergence)**.** Suppose f and $f_0, f_1, f_2, \dots$ are real-valued functions all defined on the same set I (for example an interval).

- The sequence $(f_n)_{n=0}^{\infty}$ converges to the function f **pointwise** on I if
$$\lim_{n\to\infty} f_n(x) = f(x)$$
for each $x \in I$.
[Equivalently: for each $x \in I$ and for each $\varepsilon > 0$ there is an N (which may depend on x and ε) such that $\left|f_n(x) - f(x)\right| < \varepsilon$ for all $n \geq N$.]
- The sequence $(f_n)_{n=0}^{\infty}$ converges to the function f **uniformly** on I if
$$\lim_{n\to\infty} \sup_{x\in I} \left|f_n(x) - f(x)\right| = 0.$$
[Equivalently: for each $\varepsilon > 0$ there is an N (which may depend on ε) such that $\left|f_n(x) - f(x)\right| < \varepsilon$ for all $x \in I$ and all $n \geq N$.]
- The function *series* $\sum_{n=0}^{\infty} f_n(x)$ converges pointwise/uniformly to the function $s(x)$ if the *sequence of partial sums* $s_n(x) = \sum_{k=0}^{n} f_k(x)$ converges pointwise/uniformly to $s(x)$.

Remark. The same definitions also apply to complex-valued or vector-valued functions, etc., with the suitable interpretation of what $\left|f_n(x) - f(x)\right|$ means.

Theorem. Uniform convergence implies pointwise convergence, but not the other way around.

Proof. If $f_n \to f$ uniformly, then for a given $\varepsilon > 0$ one can find an N which works for all x, so the same number N will work for each particular x in the definition of pointwise convergence.

An example showing that the converse fails is the sequence $f_n(x) = x^n$ on the interval $[0, 1]$, which converges pointwise, but *not uniformly*, to the discontinuous function
$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$
□

Theorem (The uniform limit theorem)**.** If each f_n is continuous on I, and $f_n \to f$ *uniformly*, then f is continuous on I.

Proof. Suppose $a \in I$. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, there is an N such that $\left|f_N(x) - f(x)\right| < \varepsilon/3$ for all $x \in I$. Since f_N is continuous, there is a $\delta > 0$ such that $\left|f_N(x) - f_N(a)\right| < \varepsilon/3$ for all $x \in I$ such that $|x - a| < \delta$. The triangle inequality gives
$$\begin{aligned} \left|f(x) - f(a)\right| &= \left|f(x) - f_N(x) + f_N(x) - f_N(a) + f_N(a) - f(a)\right| \\ &\leq \left|f(x) - f_N(x)\right| + \left|f_N(x) - f_N(a)\right| + \left|f_N(a) - f(a)\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$
for all $x \in I$ such that $|x - a| < \delta$. Thus f is continuous at a. □

Theorem (The Weierstrass M-test)**.** If the numerical series $\sum_{n=0}^{\infty} M_n$ converges, and if $\left|f_n(x)\right| \le M_n$ for all $x \in I$, then the function series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly (and absolutely) on I to some function $S(x)$.

Proof. (Omitted.) □

Remark. In the M-test, if each f_n is *continuous*, then the sum S is also a continuous function. This follows by applying the uniform limit theorem to the sequence of partial sums.

Equivalent integral equation

Lemma. Let $I \subseteq \mathbf{R}$ be an open interval (bounded or unbounded), and assume that $\mathbf{X}\colon I \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous. Let $t_0 \in I$. Then the function $\mathbf{x}(t)$ $(t \in I)$ is a continuously differentiable solution of the initial value problem

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{X}\big(t, \mathbf{x}(t)\big) \quad \text{for } t \in I, \\ \mathbf{x}(t_0) &= \mathbf{c}, \end{aligned} \tag{A}$$

if and only if it is a continuous solution of the integral equation

$$\mathbf{x}(t) = \mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}(s)\big)\, ds \quad \text{for } t \in I. \tag{B}$$

The same statement holds also for closed intervals I, provided that the derivative $\dot{\mathbf{x}}(t)$ is interpreted as a *one-sided* derivative when t is an *endpoint* of I.

Proof. This is an immediate consequence of the fundamental theorem of calculus. □

Picard iteration

Picard's idea for proving the existence of a solution to problem (A) is to recursively define an infinite sequence of functions

$$\mathbf{x}_0(t), \quad \mathbf{x}_1(t), \quad \mathbf{x}_2(t), \quad \ldots \qquad (t \in I)$$

by the formulas

$$\begin{aligned} \mathbf{x}_0(t) &= \mathbf{c}, \\ \mathbf{x}_n(t) &= \mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}_{n-1}(s)\big)\, ds \qquad \text{for } n \ge 1, \end{aligned}$$

and to show that this sequence converges (under some conditions) to a continuous function $\mathbf{x}(t)$ which satisfies the integral equation (B), and hence also the initial value problem (A). The uniqueness of this solution is proved by separate argument (but under the same conditions).

We may note right away that each function $\mathbf{x}_n(t)$ in the sequence is differentiable on I. This is obvious for $n = 0$ since $\mathbf{x}_0$ is just a constant function, and for $n \ge 1$ it follows from the fundamental theorem of calculus:

$$\frac{d\mathbf{x}_n}{dt}(t) = \mathbf{X}\big(t, \mathbf{x}_{n-1}(t)\big).$$

And since the functions $\mathbf{x}_n(t)$ are differentiable, they are automatically continuous as well.

The Lipschitz condition

How to prove that the sequence defined by Picard iteration converges? Answer: We write

$$\begin{aligned} \mathbf{x}_n &= \big(\mathbf{x}_n - \mathbf{x}_{n-1}\big) + \cdots + \big(\mathbf{x}_2 - \mathbf{x}_1\big) + \big(\mathbf{x}_1 - \mathbf{x}_0\big) + \mathbf{x}_0 \\ &= \mathbf{c} + \sum_{k=1}^{n} \big(\mathbf{x}_k - \mathbf{x}_{k-1}\big), \end{aligned}$$

and apply the Weierstrass M-test to show that the function series

$$\mathbf{c} + \sum_{k=1}^{\infty} \left(\mathbf{x}_k - \mathbf{x}_{k-1} \right)$$

converges (uniformly, on some interval).

For this, we will need to estimate the differences $\mathbf{x}_k - \mathbf{x}_{k-1}$. For $k \geq 2$ we have

$$\begin{aligned} \mathbf{x}_k(t) - \mathbf{x}_{k-1}(t) &= \left(\mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}_{k-1}(s)\big)\, ds \right) \\ &\quad - \left(\mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}_{k-2}(s)\big)\, ds \right) \\ &= \int_{t_0}^{t} \Big(\mathbf{X}\big(s, \mathbf{x}_{k-1}(s)\big) - \mathbf{X}\big(s, \mathbf{x}_{k-2}(s)\big) \Big)\, ds. \end{aligned}$$

To get anything interesting out of this expression, it's necessary to assume something about the function $\mathbf{X}$. The natural assumption in this context is that $\mathbf{X}$ satisfies the following so-called **Lipschitz condition** with respect to $\mathbf{x}$: there is some set $\Omega \subseteq \mathbf{R}^n$ and some constant $L > 0$ such that[1]

$$|\mathbf{X}(t,\mathbf{a}) - \mathbf{X}(t,\mathbf{b})| \leq L\,|\mathbf{a} - \mathbf{b}| \qquad \text{for all } t \in I \text{ and for all } \mathbf{a}, \mathbf{b} \in \Omega. \tag{Lip}$$

This assumption allows us to make an estimate where we get rid of the terms containing $\mathbf{X}$, as follows: if

$$\mathbf{x}_{k-1}(t) \in \Omega \quad \text{and} \quad \mathbf{x}_{k-2}(t) \in \Omega \quad \text{for all } t \in I,$$

then for $t_0 \leq t \in I$ we have

$$\begin{aligned} |\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)| &= \left| \int_{t_0}^{t} \Big(\mathbf{X}\big(s, \mathbf{x}_{k-1}(s)\big) - \mathbf{X}\big(s, \mathbf{x}_{k-2}(s)\big) \Big)\, ds \right| && \text{(put } |\cdots| \text{ around the equality above)} \\ &\leq \int_{t_0}^{t} \Big| \mathbf{X}\big(s, \mathbf{x}_{k-1}(s)\big) - \mathbf{X}\big(s, \mathbf{x}_{k-2}(s)\big) \Big|\, ds && \text{(triangle inequality for integrals)} \\ &\leq L \int_{t_0}^{t} |\mathbf{x}_{k-1}(s) - \mathbf{x}_{k-2}(s)|\, ds && \text{(because of the Lipschitz condition),} \end{aligned}$$

and similarly for $t_0 \geq t \in I$ with the bounds of integration in the opposite order:

$$|\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)| \leq L \int_{t}^{t_0} |\mathbf{x}_{k-1}(s) - \mathbf{x}_{k-2}(s)|\, ds.$$

In the proofs below, these inequalities will allow us to use knowledge about one difference $\mathbf{x}_{k-1} - \mathbf{x}_{k-2}$ to say something about the next difference $\mathbf{x}_k - \mathbf{x}_{k-1}$.

The Picard–Lindelöf theorem

Theorem (Picard–Lindelöf theorem, **global** version)**.** Let $I \subseteq \mathbf{R}$ be an open interval, and assume that $\mathbf{X}\colon I \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and satisfies the Lipschitz condition (Lip) on the whole space $\mathbf{R}^n$:

$$|\mathbf{X}(t,\mathbf{a}) - \mathbf{X}(t,\mathbf{b})| \leq L\,|\mathbf{a} - \mathbf{b}| \qquad \text{for all } t \in I \text{ and for all } \mathbf{a}, \mathbf{b} \in \mathbf{R}^n.$$

[1] For example, in the one-dimensional case, if the partial derivative $\frac{\partial X}{\partial x}(t,x)$ exists and satisfies the boundedness condition

$$\left| \frac{\partial X}{\partial x}(t,x) \right| \leq L \qquad \text{for all } t \in I \text{ and for all } a, b \in \Omega,$$

then the mean value theorem for derivatives implies that

$$|X(t,a) - X(t,b)| = \left| (b-a)\, \frac{\partial X}{\partial x}(t,\xi) \right| \leq L\,|b-a| \qquad \text{for all } t \in I \text{ and for all } a, b \in \Omega,$$

so that the Lipschitz condition holds. Similarly in higher dimensions. For simplicity, one often uses the stronger assumption that $\mathbf{X}(t,\mathbf{x})$ is of class C^1; this gives the Lipschitz condition automatically.

Then for any $t_0 \in I$ and any $\mathbf{c} \in \mathbf{R}^n$, the initial value problem (A),

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{X}\big(t, \mathbf{x}(t)\big) \quad \text{for } t \in I, \\ \mathbf{x}(t_0) &= \mathbf{c}, \end{aligned}$$

has exactly one solution $\mathbf{x}(t)$. (Note that the interval I may be bounded or unbounded, and that the solution is defined on the whole interval $t \in I$.)

Theorem (Picard–Lindelöf theorem, **local** version)**.** Let $I \subseteq \mathbf{R}$ be an open interval, and assume that $\mathbf{X}\colon I \times \Omega \to \mathbf{R}^n$ is continuous and satisfies the Lipschitz condition (Lip) on some open set $\Omega \subseteq \mathbf{R}^n$:

$$|\mathbf{X}(t, \mathbf{a}) - \mathbf{X}(t, \mathbf{b})| \le L\,|\mathbf{a} - \mathbf{b}| \qquad \text{for all } t \in I \text{ and for all } \mathbf{a}, \mathbf{b} \in \Omega.$$

Given any $t_0 \in I$ and any $\mathbf{c} \in \Omega$, take $h > 0$ and $r > 0$ small enough that the interval $J = [t_0 - h, t_0 + h]$ is contained in I and the closed ball $B = \overline{B(\mathbf{c}, r)}$ is contained in Ω, and let

$$C = \max_{(t,\mathbf{x}) \in J \times B} \big|\mathbf{X}\big(t, \mathbf{x}\big)\big|.$$

(This maximum exists by the extreme value theorem.) Then the initial value problem (A),

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{X}\big(t, \mathbf{x}(t)\big) \quad \text{for } t \in [t_0 - \varepsilon, t_0 + \varepsilon] \text{ where } \varepsilon = \min(h, r/C), \\ \mathbf{x}(t_0) &= \mathbf{c}, \end{aligned}$$

has exactly one solution $\mathbf{x}(t)$. (Note that we cannot in general guarantee that the solution is defined on the whole interval I, only on a subinterval.)

Proof of the ***global*** *version.* Define the sequence $\big(\mathbf{x}_n(t)\big)_{n=0}^{\infty}$ for $t \in I$ by Picard iteration as above.

Take any $S \in I$ and $T \in I$ with $S < t_0 < T$. We will show that there is a unique solution defined on the interval $[S, T]$, and since S and T are arbitrary, this implies that there is a unique solution on the whole interval I.

Since the function $\mathbf{X}$ is continuous and the interval $[S, T]$ is closed and bounded, the maximum

$$M = \max_{t \in [S,T]} |\mathbf{X}(t, \mathbf{c})|$$

exists, by the extreme value theorem.

Let $t \in [t_0, T]$ to begin with. Then we have

$$|\mathbf{x}_1(t) - \mathbf{x}_0(t)| = \left|\left(\mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{c}\big)\, ds\right) - \mathbf{c}\right| \le \int_{t_0}^{t} \big|\mathbf{X}\big(s, \mathbf{c}\big)\big|\, ds \le M\,(t - t_0),$$

Now that we have an estimate for the first difference $\mathbf{x}_1 - \mathbf{x}_0$, we can start estimating the other differences $\mathbf{x}_k - \mathbf{x}_{k-1}$ successively, using the inequality that we derived in the section about the Lipschitz condition above. This gives (still for $t \in [t_0, T]$):

$$\begin{aligned} |\mathbf{x}_2(t) - \mathbf{x}_1(t)| &\le L \int_{t_0}^{t} |\mathbf{x}_1(s) - \mathbf{x}_0(s)|\, ds \le L \int_{t_0}^{t} M\,(s - t_0)\, ds = \frac{LM}{2}(t - t_0)^2, \\ |\mathbf{x}_3(t) - \mathbf{x}_2(t)| &\le L \int_{t_0}^{t} |\mathbf{x}_2(s) - \mathbf{x}_1(s)|\, ds \le L \int_{t_0}^{t} \frac{L^2 M}{2}(s - t_0)^2\, ds = \frac{L^2 M}{2 \cdot 3}(t - t_0)^3, \\ |\mathbf{x}_4(t) - \mathbf{x}_3(t)| &\le L \int_{t_0}^{t} |\mathbf{x}_3(s) - \mathbf{x}_2(s)|\, ds \le L \int_{t_0}^{t} \frac{L^2 M}{2 \cdot 3}(s - t_0)^3\, ds = \frac{L^3 M}{2 \cdot 3 \cdot 4}(t - t_0)^4, \end{aligned}$$

and so on, with an obvious pattern emerging. To get a uniform estimate, let $t = T$:

$$|\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)| \le \frac{L^{k-1} M}{k!}(T - t_0)^k \qquad \text{for all } t \in [t_0, T] \text{ and } k \ge 1.$$

If we instead consider $t \in [S, t_0]$, we find in the same way that

$$|\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)| \le \frac{L^{k-1}M}{k!}(t_0 - S)^k \qquad \text{for all } t \in [S, t_0] \text{ and } k \ge 1.$$

We can combine these two estimates into a single uniform estimate over the whole interval $[S, T]$, less sharp but still good enough for our purposes:

$$|\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)| \le \frac{L^{k-1}M}{k!}(T - S)^k \qquad \text{for all } t \in [S, T] \text{ and } k \ge 1.$$

The numerical series

$$\sum_{k=1}^{\infty} \frac{L^{k-1}M}{k!}(T-S)^k = \frac{M}{L}\sum_{k=1}^{\infty} \frac{\big(L(T-S)\big)^k}{k!} = \frac{M}{L}\big(e^{L(T-S)} - 1\big)$$

converges, so the Weierstrass M-test shows the uniform convergence on $[S, T]$ of the function series that we have majorized,

$$\mathbf{c} + \sum_{k=1}^{\infty}\Big(\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)\Big).$$

The partial sums of this series are just the functions

$$\mathbf{x}_n(t) = \mathbf{c} + \sum_{k=1}^{n}\Big(\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)\Big),$$

so what we have shown is that the function sequence $\big(\mathbf{x}_n\big)_{n=0}^{\infty}$ converges uniformly on $[S, T]$ to some function $\mathbf{x}$. And since each function $\mathbf{x}_n$ is continuous, the uniform limit theorem shows that this function $\mathbf{x}$ is continuous.

Moreover, for $t \in [S, T]$,

$$\begin{aligned}
&\mathbf{x}(t) - \mathbf{c} - \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}(s)\big)\, ds \\
&= \mathbf{x}(t) - \mathbf{x}_n(t) + \mathbf{x}_n(t) - \mathbf{c} - \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}(s)\big)\, ds && \text{(add and subtract } \mathbf{x}_n) \\
&= \mathbf{x}(t) - \mathbf{x}_n(t) + \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}_{n-1}(s)\big)\, ds - \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}(s)\big)\, ds && \text{(use definition of } \mathbf{x}_n) \\
&= \mathbf{x}(t) - \mathbf{x}_n(t) + \int_{t_0}^{t} \Big(\mathbf{X}\big(s, \mathbf{x}_{n-1}(s)\big)\, ds - \mathbf{X}\big(s, \mathbf{x}(s)\big)\Big)\, ds,
\end{aligned}$$

so

$$\begin{aligned}
\left|\mathbf{x}(t) - \mathbf{c} - \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}(s)\big)\, ds\right| &\le |\mathbf{x}(t) - \mathbf{x}_n(t)| + \left|\int_{t_0}^{t} \Big|\mathbf{X}\big(s, \mathbf{x}_{n-1}(s)\big)\, ds - \mathbf{X}\big(s, \mathbf{x}(s)\big)\Big|\, ds\right| \\
&\le |\mathbf{x}(t) - \mathbf{x}_n(t)| + L\left|\int_{t_0}^{t} |\mathbf{x}_{n-1}(s) - \mathbf{x}(s)|\, ds\right|,
\end{aligned}$$

where the right-hand side can be made arbitrarily small by taking n large enough, due to the *uniform* convergence

$$\max_{t \in [S,T]} |\mathbf{x}_n(t) - \mathbf{x}(t)| \to 0 \qquad \text{as } n \to \infty.$$

Therefore, since the left-hand side is nonnegative and independent of n, it must be zero:

$$\mathbf{x}(t) - \mathbf{c} - \int_{t_0}^{t} \mathbf{X}\big(s, \mathbf{x}(s)\big)\, ds = 0.$$

In other words, the continuous function $\mathbf{x}(t)$ satisfies the integral equation (B) on the interval $[S, T]$, and thus it also satisfies the equivalent initial value problem (A) on $[S, T]$.

To show uniqueness, assume that the function $\mathbf{y}(t)$ is also a continuous solution of (B) on $[S,T]$. Then the maximum

$$A = \max_{t\in[S,T]} \left|\mathbf{y}(t) - \mathbf{c}\right|$$

exists, by the extreme value theorem. For $t \in [t_0, T]$ we first estimate

$$\begin{aligned}
\left|\mathbf{y}(t) - \mathbf{x}_1(t)\right| &= \left|\left(\mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s,\mathbf{y}(s)\big)\,ds\right) - \left(\mathbf{c} + \int_{t_0}^{t} \mathbf{X}\big(s,\mathbf{x}_0(s)\big)\,ds\right)\right| && \text{(def. of } \mathbf{y} \text{ and } \mathbf{x}_1\text{)}\\
&\le \int_{t_0}^{t} \left|\mathbf{X}\big(s,\mathbf{y}(s)\big) - \mathbf{X}\big(s,\mathbf{c}\big)\right|\,ds && \text{(triangle inequality for integrals)}\\
&\le L\int_{t_0}^{t} \left|\mathbf{y}(s) - \mathbf{c}\right|\,ds && \text{(Lipschitz condition)}\\
&\le LA(t-t_0) && \text{(definition of } A\text{),}
\end{aligned}$$

and then successively (in a manner very similar to what we did earlier)

$$\begin{aligned}
\left|\mathbf{y}(t) - \mathbf{x}_2(t)\right| &\le \frac{L^2 A(t-t_0)^2}{2},\\
\left|\mathbf{y}(t) - \mathbf{x}_3(t)\right| &\le \frac{L^3 A(t-t_0)^3}{2\cdot 3},\\
\left|\mathbf{y}(t) - \mathbf{x}_4(t)\right| &\le \frac{L^4 A(t-t_0)^4}{2\cdot 3\cdot 4},
\end{aligned}$$

and so on. Together with the similar estimates for $t \in [S, t_0]$ we get

$$\left|\mathbf{y}(t) - \mathbf{x}_n(t)\right| \le \frac{L^n A(T-S)^n}{n!} \qquad \text{for all } t \in [S,T] \text{ and } n \ge 0.$$

The right-hand side tends to zero as $n \to \infty$, so the limit of the left-hand side, $\left|\mathbf{y}(t) - \mathbf{x}(t)\right|$, must also be zero. Thus $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \in [S,T]$, and uniqueness is proved. □

*Idea of proof of the **local** version.* Do more or less the same thing, but also use the restrictions to make sure that the Picard iteration doesn't take us outside of the region where the Lipschitz condition holds. □

Exercises

- Integral equations and Picard iteration: A40.
- Non-uniqueness and non-existence: A41, A42.
- Grönwall's lemma: A43, A44.

Additional problems

A40 (a) Find the exakt solution to the integral equation

$$x(t) = 1 + \int_0^t x(s)\,ds.$$

For comparison, also compute the sequence of Picard approximations

$$x_n(t) = 1 + \int_0^t x_{n-1}(s)\,ds$$

starting with the constant function $x_0(t) = 1$. Answer.

(b) Do the same for

$$x(t) = 3 + \int_0^t 4s\,x(s)\,ds.$$

A41 Consider the ODE $t\,\dot{x} = 2x$. (Notice that the coefficient of $\dot{x}$ equals zero at $t = 0$, so we might expect some "trouble" there; it's a *singular point* of the equation.)

(a) Verify that

$$x(t) = \begin{cases} t^2, & t \ge 0, \\ Ct^2, & t < 0 \end{cases}$$

satisfies the ODE for any constant C. Thus there are infinitely many solutions satisfying the condition $x(1) = 1$.

(b) Show that there are also infinitely many solutions satisfying the condition $x(0) = 0$, but no solutions satisfying $x(0) = b$ with $b \ne 0$.

A42 Find *all* functions $x(t)$, $t \in \mathbf{R}$, which satisfy

$$\dot{x} = 2\sqrt{|x|}, \qquad x(0) = 0.$$

A43 There are several variants of **Grönwall's lemma** (or **Grönwall's inequality**), which all boil down to the fact that if a function satisfies a differential or integral *inequality* of a certain form, then it can be no bigger than the solution of the corresponding differential or integral *equation*.

(The inequality limits how fast the function can grow, and to push those limits to the maximum and *grow as fast as possible*, the function should satisfy the inequality *with equality*.)

Here your task is to prove (with guidance) the following version of Grönwall's lemma:

Theorem. Let $a(t)$ and $b(t)$ be continuous functions with $b(t) \ge 0$, and suppose that $u(t)$ satisfies the integral inequality

$$u(t) \le a(t) + \int_0^t b(s)\,u(s)\,ds \qquad \text{for } t \ge 0.$$

Then

$$u(t) \le y(t) \qquad \text{for } t \ge 0,$$

where $y(t)$ is the solution of the corresponding integral equation

$$y(t) = a(t) + \int_0^t b(s)\,y(s)\,ds,$$

namely

$$y(t) = a(t) + \int_0^t a(s)\,b(s)\,e^{B(t)-B(s)}\,ds, \qquad \text{where} \quad B(t) = \int_0^t b(\tau)\,d\tau.$$

Outline of proof. Follow these steps:

- First rewrite the integral equation for $y(t)$ as an ODE for the integral $I(t) = \int_0^t b(s)\,y(s)\,ds = y(t) - a(t)$ appearing on the right-hand side:

$$I'(t) - b(t)\,I(t) = a(t)\,b(t), \qquad I(0) = 0.$$

- Multiply both sides by the integrating factor $e^{-B(t)}$ and integrate from 0 to t.
- After having solved for $I(t)$ in this way, you also know what $y(t)$ is, namely $y(t) = a(t) + I(t)$. Verify that your expression for $y(t)$ agrees with the formula for $y(t)$ given in the theorem above.
- Next consider the inequality for $u(t)$. Let $J(t) = \int_0^t b(s)\,u(s)\,ds$ be the integral appearing on the right-hand side. It satisfies $J'(t) = b(t)\,u(t)$. By assumption we have $u \le a + J$ and $b \ge 0$, and therefore $J' = bu \le b(a + J)$. So we know that J satisfies

$$J'(t) - b(t)\,J(t) \le a(t)\,b(t), \qquad J(0) = 0.$$

- Convince yourself that you can now perform exactly the same steps as when you solved the equation for $I(t)$ (multiply by the integrating factor, integrate, etc.), but with J instead of I and "≤" instead of "=". (Here it's important that $t \geq 0$.)
- And since you do the same steps, you will get the same result in the end, except with "$u(t) \leq$" instead of "$y(t) =$".
- This proves that $u(t) \leq y(t)$ for $t \geq 0$. Done! □

As a bonus question, see if you can prove this variant of Grönwall's lemma in a similar way:

Theorem. Let g be a continuous function (which need not be positive). If $u(t)$ is continuous for $t \geq 0$ and differentiable for $t > 0$, and satisfies the differential inequality

$$\begin{aligned} u'(t) &\leq g(t)\,u(t) \qquad \text{for } t > 0,\\ u(0) &= c, \end{aligned}$$

then

$$u(t) \leq y(t) \qquad \text{for } t \geq 0,$$

where $y(t)$ is the solution of the corresponding differential equation

$$\begin{aligned} y'(t) &= g(t)\,y(t) \qquad \text{for } t > 0,\\ y(0) &= c, \end{aligned}$$

namely

$$y(t) = c\,e^{G(t)}, \qquad G(t) = \int_0^t g(s)\,ds.$$

A44 (a) Compute the solution $x(t)$ of the initial value problem $\dot{x} = x^2$, $x(0) = c > 0$, and note that it blows up after finite time (as $t \nearrow 1/c$).

(b) In part (a), the right-hand side of the ODE was obviously quadratic in x.

In contrast, show that if the right-hand side of the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is **linearly bounded**, meaning that there are constants $a \geq 0$ and $b \geq 0$ such that

$$|\mathbf{X}(\mathbf{x})| \leq a + b|\mathbf{x}| \quad \text{for all } \mathbf{x} \in \mathbf{R}^n,$$

then the solution $\mathbf{x}(t)$ with initial value $\mathbf{x}(0) = \mathbf{x}_0$ cannot blow up in finite time, so it is defined for all $t \geq 0$ (regardless of $\mathbf{x}_0$).

(We are assuming that the vector field $\mathbf{X}(\mathbf{x})$ is nice enough for the flow to exist at least locally, say of class C^1. It can be proved that if $\mathbf{X}(\mathbf{x})$ is defined for all $\mathbf{x} \in \mathbf{R}^n$, then the only way for solutions to cease existing after finite time is that $|\mathbf{x}| \to \infty$. Let us take this fact for granted here.) Hint.

Lecture 10. Linear equations with non-constant coefficients

(Not covered in Arrowsmith & Place; see notes below instead.)

Second-order linear ODEs

Many ODE books with a more "classical" flavour allocate plenty of space to the topic of second order (inhomogeneous) linear ODES,

$$\ddot{x} + p_1(t)\,\dot{x} + p_0(t)\,x = f(t),$$

which appear in many applications, and are also of historical importance. Some particular such ODEs have been studied so much that one could easily spend several courses on them alone, like the **Bessel equation** (with parameter $\alpha \in \mathbf{C}$),

$$\ddot{x} + \tfrac{1}{t}\,\dot{x} + \tfrac{t^2-\alpha^2}{t^2}\,x = 0.$$

Any second-order ODE can be rewritten as a system of two first-order ODEs, for example by letting $x_1 = x$ and $x_2 = \dot{x}$:

$$\ddot{x}+p_1(t)\,\dot{x}+p_0(t)\,x = f(t) \quad\Longleftrightarrow\quad \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_2 \\ f(t)-p_1(t)\,x_2-p_0(t)\,x_1\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p_0(t) & -p_1(t)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\ f(t)\end{pmatrix}.$$

Here we will focus more on systems of first-order (inhomogeneous) linear ODEs in general:

$$\dot{\mathbf{x}} = A(t)\,\mathbf{x}+\mathbf{f}(t),$$

where $\mathbf{x}(t) \in \mathbf{R}^n$ and $A(t)$ is some $n\times n$ matrix. From the general results about systems we can then obtain some of the basic facts about second-order linear ODEs simply by considering the special case when $n = 2$ and $\dot{x}_1 = x_2$.

First-order systems of linear ODEs

Theorem. Assume that $A(t)$ and $\mathbf{f}(t)$ are *continuous* on some time interval I (bounded or not) containing $t = 0$. Then the initial value problem

$$\dot{\mathbf{x}} = A(t)\,\mathbf{x}+\mathbf{f}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

has a unique solution defined on all of I.

Proof. Let $J = [t_1, t_2]$ be an arbitrary compact subinterval of I containing $t = 0$. By the extreme value theorem, each matrix entry $A_{ij}(t)$ is bounded on J, and since there are only finitely many matrix entries, there must be a common constant which bounds all of them:

$$\left|A_{ij}(t)\right| \le C \qquad \text{for all } i \text{ and } j\text{, and all } t \in J.$$

Then the ith entry in the matrix product $A(t)\,\mathbf{y}$ can be estimated using the triangle inequality and the Cauchy–Schwarz inequality:

$$\left|\big(A(t)\,\mathbf{y}\big)_i\right| = \left|A_{i1}(t)\,y_1+\cdots+A_{in}(t)\,y_n\right| \le C\left|y_1\right|+\cdots+C\left|y_n\right| \le C\sqrt{n}\left|\mathbf{y}\right|.$$

Squaring, summing over i, and taking the square root, we get

$$\left|A(t)\,\mathbf{y}\right| \le Cn\left|\mathbf{y}\right|,$$

for any vector $\mathbf{y}$.

Using this property we can show that the right-hand side $\mathbf{X}(t,\mathbf{x}) = A(t)\,\mathbf{x}+\mathbf{f}(t)$ satisfies the global Lipschitz condition with Lipschitz constant $L = Cn$. Indeed,

$$\mathbf{X}(t,\mathbf{a})-\mathbf{X}(t,\mathbf{b}) = \Big(A(t)\,\mathbf{a}+\mathbf{f}(t)\Big)-\Big(A(t)\,\mathbf{b}+\mathbf{f}(t)\Big) = A(t)\,\mathbf{a}-A(t)\,\mathbf{b} = A(t)\,(\mathbf{a}-\mathbf{b})$$

implies that

$$|\mathbf{X}(t,\mathbf{a})-\mathbf{X}(t,\mathbf{b})| = |A(t)(\mathbf{a}-\mathbf{b})| \le Cn\,|\mathbf{a}-\mathbf{b}| \qquad \text{for all } t \in J \text{ and for all } \mathbf{a}, \mathbf{b} \in \mathbf{R}^n.$$

By the global Picard–Lindelöf theorem there is therefore a unique solution defined on all of J, and since the subinterval J was arbitrary, we can extend this solution as far as we like inside I. □

Theorem. The general solution of the inhomogeneous system $\dot{\mathbf{x}}-A(t)\,\mathbf{x} = \mathbf{f}(t)$ has the form

$$\mathbf{x}(t) = \mathbf{x}_{\text{hom}}(t)+\mathbf{x}_{\text{part}}(t),$$

where $\mathbf{x}_{\text{part}}$ is some particular solution, and $\mathbf{x}_{\text{hom}}$ is the general solution of the corresponding homogeneous system $\dot{\mathbf{x}}-A(t)\,\mathbf{x} = \mathbf{0}$.

Proof. This is just a fact about linearity, and not really about differential equations. If $\mathbf{x} = \mathbf{x}_1$ and $\mathbf{x} = \mathbf{x}_2$ are two particular solutions, then by linearity their difference $\mathbf{x} = \mathbf{x}_1-\mathbf{x}_2$ satisfies the homogeneous system. □

The homogeneous case

Let's look at the homogeneous system $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$ first. If $A(t) = A$ is a *constant* matrix, we have seen before that the solution is simply

$$\mathbf{x}(t) = e^{tA}\,\mathbf{x}(0),$$

but if $A(t)$ is *time-dependent*, we can't in general find an explicit solution formula like that. But we can still say a few things in principle about the structure of the solution. To formulate the theorem, let

$$(\mathbf{e}_1, \dots, \mathbf{e}_n)$$

denote the standard basis for $\mathbf{R}^n$, i.e., $\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots, 0)^T$ with a 1 in the kth position.

Theorem (Solution space)**.** The homogeneous system $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$ has an n-dimensional solution space. For example, a basis is given by the functions $\mathbf{g}_1(t), \dots, \mathbf{g}_n(t)$, where $\mathbf{x}(t) = \mathbf{g}_k(t)$ is the (unique) solution of the initial value problem starting at $\mathbf{e}_k$:

$$\dot{\mathbf{x}} = A(t)\,\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{e}_k.$$

In terms of the $n \times n$-matrix

$$\Phi(t) = \Big[\mathbf{g}_1(t), \dots, \mathbf{g}_n(t)\Big],$$

any solution has the form

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0). \tag{SOL}$$

Proof. Let $\mathbf{x}(t) = \mathbf{z}(t)$ be any solution of $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$, and let $\mathbf{c} = \mathbf{z}(0)$. Then the functions $\mathbf{x}(t) = \mathbf{z}(t)$ and

$$\mathbf{x}(t) = c_1\,\mathbf{g}_1(t) + \cdots + c_n\,\mathbf{g}_n(t)$$

both satisfy the initial value problem

$$\dot{\mathbf{x}} = A(t)\,\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{c},$$

so by uniqueness they must be the same function:

$$\mathbf{z}(t) = c_1\,\mathbf{g}_1(t) + \cdots + c_n\,\mathbf{g}_n(t) = \Phi(t)\,\mathbf{c}.$$

Thus an arbitrary solution $\mathbf{z}$ can be written as a linear combination of the functions $\mathbf{g}_k$, which shows that they span the solution space, and the formula (SOL) also follows.

To show that the functions $\mathbf{g}_k$ are linearly independent, suppose that the linear combination $\mathbf{x}(t) = \sum c_k\,\mathbf{g}_k(t)$ is the zero function; then in particular it's zero when $t = 0$:

$$\mathbf{0} = \mathbf{x}(0) = \sum c_k\,\mathbf{g}_k(0) = \sum c_k\,\mathbf{e}_k = (c_1, \dots, c_n)^T.$$

In other words, $c_1 = \cdots = c_n = 0$. □

Proposition. The columns of an $n \times n$ matrix $\Phi(t)$ are solutions of the linear system $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$ if and only if the matrix itself is a solution of the matrix-valued linear ODE

$$\dot{\Phi}(t) = A(t)\,\Phi(t).$$

Proof. This is an immediate consequence of how matrix multiplication works: the kth column in $A\Phi$ equals A times the kth column in Φ. □

Definition. Any time-dependent $n \times n$ matrix whose columns are *linearly independent* solutions of $\dot{\mathbf{x}} = A(t)\,\mathbf{x}$ is called a **fundamental matrix** for the system.

Remark. The matrix $\Phi(t)$ in the theorem above is thus one particular fundamental matrix, distinguished by the property that $\Phi(0) = I$. Any other fundamental matrix $\Psi(t)$ has the form $\Psi(t) = \Phi(t)\,M$ where $M = \Psi(0)$ is a nonsingular constant $n \times n$ matrix; indeed, this is just a change of basis in the solution space. In terms of such a Ψ, the general solution of the initial value problem is $\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0) = \Psi(t)\,M^{-1}\,\mathbf{x}(0)$.

Rephrasing what we said before the theorem: If A is *constant*, then $\Phi(t) = e^{tA}$ is the fundamental matrix satisfying $\Phi(0) = I$, but if A is *time-dependent* we cannot in general compute the fundamental matrix $\Phi(t)$ explicitly; we only know from the Picard–Lindelöf theorem that it exists and is unique. But curiously enough, we can always compute its determinant:

Theorem (Liouville's identity)**.** If $\Phi(t)$ is a solution of the matrix ODE $\dot\Phi(t) = A(t)\Phi(t)$, then its determinant[1]

$$W(t) = \det\Phi(t)$$

satisfies the scalar ODE

$$\dot W(t) = \operatorname{tr}\big(A(t)\big)\,W(t),$$

and hence

$$W(t) = \exp\left(\int_0^t \operatorname{tr}\big(A(s)\big)\,ds\right) W(0).$$

Proof. A down-to-earth way of proving this is to just compute. Consider the 3×3 case, for ease of notation, so that

$$W(x) = \det\big(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big) = \begin{vmatrix} g_{11} & g_{21} & g_{31} \\ g_{12} & g_{22} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} = g_{11}g_{22}g_{33} + \cdots.$$

When differentiating this using the product rule for derivatives, each of the $n!$ terms in the determinant will give rise to n terms, and the resulting sum can be rearranged back into a sum of n determinants:

$$\begin{aligned} \dot W &= \frac{d}{dt}\det\big(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big) \\ &= \left(\frac{dg_{11}}{dt} g_{22}g_{33} + g_{11}\frac{dg_{22}}{dt} g_{33} + g_{11}g_{22}\frac{dg_{33}}{dt}\right) + \cdots \\ &= \det\left(\frac{d\mathbf{g}_1}{dt}, \mathbf{g}_2, \mathbf{g}_3\right) + \det\left(\mathbf{g}_1, \frac{d\mathbf{g}_2}{dt}, \mathbf{g}_3\right) + \det\left(\mathbf{g}_1, \mathbf{g}_2, \frac{d\mathbf{g}_3}{dt}\right) \\ &= \det\big(A\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big) + \det\big(\mathbf{g}_1, A\mathbf{g}_2, \mathbf{g}_3\big) + \det\big(\mathbf{g}_1, \mathbf{g}_2, A\mathbf{g}_3\big). \end{aligned}$$

If we expand all these determinants, what terms would we get that contain A_{11}? Answer: the terms appearing in the expression

$$\begin{vmatrix} A_{11}g_{11} & 0 & 0 \\ 0 & g_{22} & g_{32} \\ 0 & g_{23} & g_{33} \end{vmatrix} + \begin{vmatrix} 0 & A_{11}g_{21} & 0 \\ g_{12} & 0 & g_{32} \\ g_{13} & 0 & g_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & A_{11}g_{31} \\ g_{12} & g_{22} & 0 \\ g_{13} & g_{23} & 0 \end{vmatrix} = A_{11}\det\big(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big).$$

And the terms that contain A_{12} are

$$\begin{vmatrix} A_{12}g_{12} & 0 & 0 \\ 0 & g_{22} & g_{32} \\ 0 & g_{23} & g_{33} \end{vmatrix} + \begin{vmatrix} 0 & A_{12}g_{22} & 0 \\ g_{12} & 0 & g_{32} \\ g_{13} & 0 & g_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & A_{12}g_{32} \\ g_{12} & g_{22} & 0 \\ g_{13} & g_{23} & 0 \end{vmatrix} = A_{12}\begin{vmatrix} g_{12} & g_{22} & g_{32} \\ g_{12} & g_{22} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{vmatrix},$$

but this is zero since two rows are equal, so there will be no terms containing A_{12} in the expansion. Similarly, we get contributions $A_{22}\det\big(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big)$ and $A_{33}\det\big(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big)$, but no terms containing an A_{ij} with $i \neq j$. Thus,

$$\dot W = (A_{11} + A_{22} + A_{33})\det\big(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\big) = \operatorname{tr}(A)\,W,$$

as desired. The formula for $W(t)$ is obtained by solving this ODE for W using an integrating factor. □

Remark. Liouville's identity shows that if the columns of the matrix $\Phi(t)$ are *solutions of some linear system* $\dot{\mathbf{x}} = A(t)\mathbf{x}$, then $W(t) = \det\Phi(t)$ is either **identically zero** or **never zero**. The case $W(t) = 0$ occurs when the columns of Φ are linearly dependent functions, and $W(t) \neq 0$ when they are linearly independent. But beware that if we just look at the determinant of some arbitrary time-dependent matrix $\Phi(t)$, then the question of linear independence of the columns is not this simple! (See problem A49.)

[1] Called **the Wronskian determinant**, or simply **the Wronskian**.

Consider the special case of second-order linear homogeneous ODEs

$$\ddot{x} + p_1(t)\,\dot{x} + p_0(t)\,x = 0,$$

which can be written as $\dot{\mathbf{x}} = A\mathbf{x}$ with

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -p_0(t) & -p_1(t) \end{pmatrix},$$

as explained a few pages ago. Since $\operatorname{tr}\big(A(t)\big) = -p_1(t)$ in this case, Liouville's identity takes the following form:

Theorem (Abel's identity)**.** If $x = y(t)$ and $x = z(t)$ are two solutions of $\ddot{x} + p_1\dot{x} + p_0 x = 0$, then their Wronskian

$$W(t) = \begin{vmatrix} y(t) & z(t) \\ \dot{y}(t) & \dot{z}(t) \end{vmatrix} = y(t)\,\dot{z}(t) - \dot{y}(t)\,z(t)$$

satisfies

$$\dot{W}(t) = -p_1(t)\,W(t),$$

and hence

$$W(t) = \exp\left(-\int_0^t p_1(s)\,ds\right) W(0).$$

The inhomogeneous case

Now to the question of finding a **particular solution** $\mathbf{x}_{\text{part}}(t)$ of the *inhomogeneous* system

$$\dot{\mathbf{x}} - A(t)\,\mathbf{x} = \mathbf{f}(t),$$

supposing that we already know the general solution $\mathbf{x}_{\text{hom}}(t)$ of the *homogeneous* equation. (In other words, supposing that we know a fundamental matrix.) This can be done by a method called **variation of constants**, **variation of parameters** or **Lagrange's method**, as follows:

Theorem (Variation of constants)**.** If $\Phi(t)$ is a fundamental matrix for the homogeneous system, then

$$\mathbf{x}_{\text{part}}(t) = \Phi(t) \int_0^t \Phi(s)^{-1}\,\mathbf{f}(s)\,ds$$

is a particular solution of the inhomogeneous system.

Proof. The fundamental matrix satisfies $\dot{\Phi} = A\Phi$ (by definition). Make the change of variables

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t).$$

When we substitute this into the system, together with $\dot{\mathbf{x}} = \dot{\Phi}\,\mathbf{y} + \Phi\,\dot{\mathbf{y}}$, we obtain

$$\begin{aligned}
& & \dot{\mathbf{x}}(t) - A(t)\,\mathbf{x}(t) &= \mathbf{f}(t) \\
&\iff & \dot{\Phi}(t)\,\mathbf{y}(t) + \Phi(t)\,\dot{\mathbf{y}}(t) - A(t)\,\Phi(t)\,\mathbf{y}(t) &= \mathbf{f}(t) \\
&\iff & \underbrace{\Big(\dot{\Phi}(t) - A(t)\,\Phi(t)\Big)}_{=0}\,\mathbf{y}(t) + \Phi(t)\,\dot{\mathbf{y}}(t) &= \mathbf{f}(t) \\
&\iff & \Phi(t)\,\dot{\mathbf{y}}(t) &= \mathbf{f}(t) \\
&\iff & \dot{\mathbf{y}}(t) &= \Phi(t)^{-1}\,\mathbf{f}(t).
\end{aligned}$$

(We know that $\Phi(t)^{-1}$ exists for every t, since the Wronskian $W(t) = \det\Phi(t)$ is never zero, by Liouville's identity.) Now just change t to s, integrate both sides from (say) 0 to t to find a particular $\mathbf{y}(t)$ which works, and go back to the old variables $\mathbf{x}$. Done! □

Remark. There is no need to memorize the solution formula; just repeat the procedure in the proof every time you need it! The reason for the funny name "variation of constants" is that the general solution of the homogeneous system is

$$\mathbf{x}_{\text{hom}}(t) = \Phi(t)\,\mathbf{c},$$

where $\mathbf{c} = (c_1,\dots,c_n)^T$ is an arbitrary *constant* vector, and the idea here is to try to find a particular solution by "letting the constants c_k vary", i.e., by replacing them with the *time-dependent* quantities $\mathbf{y}(t) = \big(y_1(t),\dots,y_n(t)\big)^T$:

$$\mathbf{x}_{\text{part}}(t) = \Phi(t)\,\mathbf{y}(t).$$

Back to single second-order (or higher-order) ODEs

Now consider again a single linear ODE of order n, with time-dependent coefficients:

$$\frac{d^n x}{dt^n} + p_{n-1}(t)\,\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\,\frac{dx}{dt} + p_0(t)\,x = f(t).$$

The general solution of this inhomogeneous equation equals one particular solution plus the general solution of the homogeneous equation

$$\frac{d^n x}{dt^n} + p_{n-1}(t)\,\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\,\frac{dx}{dt} + p_0(t)\,x = 0.$$

The first question is how we may find a basis for the n-dimensional solution space of the homogeneous equation. Sometimes we might get lucky and find one solution (by inspired guessing or power series methods or something else). This can then be used for finding *other* solutions, by the following simple trick:

Theorem (Reduction of order)**.** Suppose $x_0(t)$ is a known solution of the homogeneous equation. Then the substitution $x(t) = Y(t)\,x_0(t)$ leads to a homogeneous equation of order $n-1$ for $y(t) = \dot Y(t)$.

Proof. Substitute

$$\begin{aligned} x &= Y x_0, \\ \dot x &= \dot Y x_0 + Y \dot x_0, \\ \ddot x &= \ddot Y x_0 + 2\dot Y \dot x_0 + Y \ddot x_0, \\ &\ \vdots \end{aligned}$$

into the homogeneous ODE for x. Then the coefficient of Y will be $x_0^{(n)} + p_{n-1}\,x_0^{(n-1)} + \cdots + p_1\,\dot x_0 + p_0\,x_0$, which equals zero by assumption. Thus only $\dot Y, \ddot Y, \dots, Y^{(n)}$ appear in the equation, so if we let $y = \dot Y$ we get an equation involving only $y, \dot y, \dots, y^{(n-1)}$. □

Remark. The reduced equation always has the trivial solution $y(t) = 0$, which gives $Y(t) = C$. But this is quite uninteresting, since $x(t) = Y(t)\,x_0(t) = C x_0(t)$ is then just a constant multiple of the already known solution $x_0(t)$.

Remark. If we manage to find a solution of a *second-order* homogeneous linear ODE, then reduction of order gives a *first-order* equation, which means that we can find a nontrivial solution with the help of an integrating factor. We can then integrate this solution $y(t)$ to find a non-constant $Y(t)$, and hence find a second *linearly independent* solution $x(t) = Y(t)\,x_0(t)$. So in this case $x_0(t)$ and $Y(t)\,x_0(t)$ will be a basis of the solution space.

For finding a particular solution of the inhomogeneous equation, we have the method of **variation of constants**, as a special case of what we did for systems. Rewrite the ODE as a first-order system by letting $x_1 = x$, $x_2 = \dot x$, etc.:

$$\begin{pmatrix} \dot x_1 \\ \dot x_2 \\ \vdots \\ \dot x_{n-1} \\ \dot x_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t) - \sum_{k=1}^n p_{k-1}(t)\,x_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -p_0(t) & -p_1(t) & \cdots & -p_{n-2}(t) & -p_{n-1}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}$$

Let's look at the case $n = 3$ to simplify notation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ f(t) - p_0(t)\,x_1 - p_1(t)\,x_2 - p_2(t)\,x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f(t) \end{pmatrix}$$

The method requires that we already know the general solution x_{hom} of the homogeneous equation, say

$$x_{\text{hom}}(t) = c_1 g_1(t) + c_2 g_2(t) + c_3 g_3(t)$$

where (g_1, g_2, g_3) is a known basis for the solution space. We are going to seek a particular solution by "letting the constants c_k vary", i.e., replacing c_k by $y_k(t)$:

$$x_{\text{part}}(t) = y_1(t)\,g_1(t) + y_2(t)\,g_2(t) + y_3(t)\,g_3(t).$$

In terms of the first-order system, this means that

$$\Phi(t) = \begin{pmatrix} g_1 & g_2 & g_3 \\ \dot{g}_1 & \dot{g}_2 & \dot{g}_3 \\ \ddot{g}_1 & \ddot{g}_2 & \ddot{g}_3 \end{pmatrix}$$

is a fundamental matrix, and we are making the substitution $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$. Simply remembering what we did for systems above, we know that this leads to $\Phi(t)\,\dot{\mathbf{y}} = \mathbf{f}(t)$:

$$\begin{pmatrix} g_1 & g_2 & g_3 \\ \dot{g}_1 & \dot{g}_2 & \dot{g}_3 \\ \ddot{g}_1 & \ddot{g}_2 & \ddot{g}_3 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f(t) \end{pmatrix}.$$

This system of equations determines $\dot{\mathbf{y}}$, which we can then integrate to find $\mathbf{y}$, and hence x_{part}.

Remark. Books which deal only with single higher-order ODEs (rather than systems of first-order ODEs) usually present the method of variation of constants in the following way, which in my opinion is rather obscure. Start with

$$x = y_1 g_1 + y_2 g_2 + y_3 g_3.$$

Take the first derivative:

$$\dot{x} = (\dot{y}_1 g_1 + \dot{y}_2 g_2 + \dot{y}_3 g_3) + (y_1 \dot{g}_1 + y_2 \dot{g}_2 + y_3 \dot{g}_3).$$

For some mysterious reason ("just because it works"), we require the first bracket to be zero: $\dot{y}_1 g_1 + \dot{y}_2 g_2 + \dot{y}_3 g_3 = 0$. So only the second bracket remains in $\dot{x} = y_1 \dot{g}_1 + y_2 \dot{g}_2 + y_3 \dot{g}_3$, and taking the derivative of this gives

$$\ddot{x} = (\dot{y}_1 \dot{g}_1 + \dot{y}_2 \dot{g}_2 + \dot{y}_3 \dot{g}_3) + (y_1 \ddot{g}_1 + y_2 \ddot{g}_2 + y_3 \ddot{g}_3).$$

Again we require the first bracket to be zero: $\dot{y}_1 \dot{g}_1 + \dot{y}_2 \dot{g}_2 + \dot{y}_3 \dot{g}_3 = 0$. This leaves $\ddot{x} = y_1 \ddot{g}_1 + y_2 \ddot{g}_2 + y_3 \ddot{g}_3$, so

$$\dddot{x} = (\dot{y}_1 \ddot{g}_1 + \dot{y}_2 \ddot{g}_2 + \dot{y}_3 \ddot{g}_3) + (y_1 \dddot{g}_1 + y_2 \dddot{g}_2 + y_3 \dddot{g}_3)$$

Now using that g_1, g_2 and g_3 satisfy the homogeneous equation, we find after some computation that if we want x to satisfy the inhomogeneous equation, then we must require the first bracket here to satisfy $\dot{y}_1 \ddot{g}_1 + \dot{y}_2 \ddot{g}_2 + \dot{y}_3 \ddot{g}_3 = f$. These three requirements are exactly the matrix equation $\Phi \dot{\mathbf{y}} = \mathbf{f}$ that we found above in a much simpler way, using systems and matrix algebra.

Exercises

- Reduction of order: A45, A46.
- Variation of constants: A47, A48.
- Wronskians: A49.

[Table of contents]

Additional problems

A45 Determine α^2 such that that the Bessel equation

$$t^2\ddot{x} + t\,\dot{x} + (t^2 - \alpha^2)\,x = 0 \qquad (t > 0)$$

has a solution $x_0(t) = t^{-1/2}\sin t$, and use reduction of order to find another (linearly independent) solution. Answer.

A46 (a) The usual rule based on the characteristic polynomial says that $x(t) = Ae^{2t} + Be^{3t}$ is the general solution of $\ddot{x} - 5\dot{x} + 6x = 0$. Give a direct proof that this *really* is the most general solution, by applying reduction of order using the known solution $x_0(t) = e^{2t}$.

(b) Similarly, show that $x(t) = (At + B)e^{-3t}$ is the general solution of $\ddot{x} + 6\dot{x} + 9x = 0$, by applying reduction of order with $x_0(t) = e^{-3t}$.

(If you look back in your calculus textbook, you will probably find arguments like these in the section explaining the theory behind the characteristic polynomial.)

A47 Find the general solution. (Variation of constants is useful for finding a particular solution, at least in the more difficult cases. But just for practice, you can use it in the simpler cases as well. And if you don't get enough of this, you can look back at problem A3.)

(a) $\ddot{x} + 2\dot{x} + x = 2\sin t$.

(b) $\ddot{x} + 9x = \cos 3t$.

(c) $\ddot{x} + x = \tan t$.

(d) $\dot{x}_1 = 2x_1 - 5x_2 + 4t$, $\dot{x}_2 = x_1 - 2x_2 + 1$.

(e) $(t^2 - 1)\ddot{x} - 2t\dot{x} + 2x = t^2 - 1$. (To find $x_{\text{hom}}(t)$, try a power series solution.)

Answers.

A48 Find the general solution of the system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 4\\ -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \frac{e^{2t}}{1+t^2}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

Answer.

A49 (a) Show that there is no second-order ODE $\ddot{x} + p_1(t)\,\dot{x} + p_0(t)\,x = 0$ whose general solution has the form $x(t) = At + B\cos t$, if we require the coefficients p_0 and p_1 to be defined and continuous for all $t \in \mathbf{R}$.

(b) Show that there is such an ODE if we remove the requirement that p_0 and p_1 be defined on the whole real line.

(c) Find a 2×2 matrix $\Phi(t)$ such that its columns are linearly independent (as functions of t), but $\det\Phi(t)$ is zero at some points.

(d) Show that things can be even worse than in (b): the functions $y(t) = t^3$ and $z(t) = |t|^3$ ($t \in \mathbf{R}$) are linearly independent, but their Wronskian is *identically* zero.

Hints.

Lesson 4

Lesson 5

These last two lessons are for catching up and working on the homework problems.

There is an optional video lecture, **Outlook: Poincaré maps, attractors, chaotic systems**, that you can watch if you like. This material is not a part of the course, but is provided as "edutainment", and to give a rough idea of some additional topics which are important in the theory of dynamical systems.

[Table of contents]

Answers, hints, solutions

A1 (a) $\dot{x} = 2x \iff (\dot{x} - 2x)e^{-2t} = 0 \iff \frac{d}{dt}\big(xe^{-2t}\big) = 0 \iff xe^{-2t} = C \iff x = Ce^{2t}$ (where C of course denotes an arbitrary real constant).

(b) $\dot{x} = 2x+7 \iff (\dot{x}-2x)e^{-2t} = 7e^{-2t} \iff \frac{d}{dt}\big(xe^{-2t}\big) = 7e^{-2t} \iff xe^{-2t} = -\frac{7}{2}e^{-2t}+C \iff x = \big(-\frac{7}{2}e^{-2t}+C\big)e^{2t} = -\frac{7}{2}+Ce^{2t}$.

(c) $\dot{x} = 2x + e^{5t} \iff (\dot{x} - 2x)e^{-2t} = e^{5t}e^{-2t} \iff \frac{d}{dt}\big(xe^{-2t}\big) = e^{3t} \iff xe^{-2t} = \frac{1}{3}e^{3t} + C \iff x = \big(\frac{1}{3}e^{3t}+C\big)e^{2t} = \frac{1}{3}e^{5t} + Ce^{2t}$.

(d) $\dot{x} = 2x + t^2e^{5t} \iff \frac{d}{dt}\big(xe^{-2t}\big) = t^2e^{3t} \iff xe^{-2t} = \int e^{3t}\cdot t^2\,dt = \frac{1}{3}e^{3t}\cdot t^2 - \frac{1}{9}e^{3t}\cdot 2t + \frac{1}{27}e^{3t}\cdot 2 + C \iff x = \big(\frac{1}{3}t^2 - \frac{2}{9}t + \frac{1}{27}\big)e^{5t} + Ce^{2t}$.

(e) $\dot{x} = 2x + t^2e^{2t} \iff (\dot{x} - 2x)e^{-2t} = t^2 \iff \frac{d}{dt}\big(xe^{-2t}\big) = t^2 \iff xe^{-2t} = \frac{1}{3}t^3 + C \iff x = \big(\frac{1}{3}t^3 + C\big)e^{2t}$.

(f) $\dot{x} = tx \iff (\dot{x} - tx)e^{-t^2/2} = 0 \iff \frac{d}{dt}\big(xe^{-t^2/2}\big) = 0 \iff xe^{-t^2/2} = C \iff x = Ce^{t^2/2}$.

(g) $\dot{x}+2tx = t \iff \frac{d}{dt}\big(xe^{t^2}\big) = te^{t^2} \iff xe^{t^2} = \frac{1}{2}e^{t^2} + C \iff x = \frac{1}{2} + Ce^{-t^2}$.

(h) For $t > 0$ we have $t\dot{x} + 2x = \sin t \iff \dot{x} + \frac{2}{t}x = \frac{\sin t}{t} \iff \big(\dot{x} + \frac{2}{t}x\big)t^2 = \frac{\sin t}{t}\cdot t^2 \iff \frac{d}{dt}\big(xt^2\big) = t\sin t \iff xt^2 = \int \sin t\cdot t\,dt = (-\cos t)\cdot t - (-\sin t)\cdot 1 + C \iff x = (\sin t - t\cos t + C)/t^2$.

(i) $(1+t^2)\dot{x}+2tx = 2t \iff \frac{d}{dt}\big((1+t^2)x\big) = 2t \iff (1+t^2)x = t^2+C \iff x = \frac{t^2+C}{1+t^2} = \big[\text{let } C = 1+D\big] = 1 + \frac{D}{1+t^2}$.

A2 (a) The constant function $x = 0$ is a solution. All other solutions are nonzero everywhere (thanks to the theorem about uniqueness of solutions, which applies since the right-hand side $X(x,t) = 2x$ is obviously a C^1 function). So when we seek these other solutions, it's safe to divide by x, which gives $\frac{1}{x}\dot{x} = 2$. Integrating both sides with respect to t yields $\ln|x| = 2t + C$, which is equivalent to $|x| = e^{t+C}$, which in turn is equivalent to $x = \pm e^{t+C} = \pm e^Ce^t$. Here C is an arbitrary real constant, which means that $D = \pm e^C$ is an arbitrary **nonzero** real constant. So the nonzero solutions are given by $x = De^t$, where $D \neq 0$. Together with the constant solution that we found to begin with, we have thus found the answer to be "$x = 0$ or $x = De^t$, $D \neq 0$". However, a simpler way to express this just "$x = De^t$, $D \in \mathbf{R}$". (You do agree that this is indeed the same thing, don't you?)

So we find (of course) the same answer as in A1a above (except for the irrelevant detail that the constant is called D instead of C), but notice how much simpler it was to use an integrating factor instead!

(b) $x = 0$ is a solution. All other solutions are given by $\frac{1}{x}\dot{x} = t \iff \ln|x| = \frac{1}{2}t^2 + C \iff x = \pm e^Ce^{t^2/2} = De^{t^2/2}$, where $D = \pm e^C \neq 0$. We can summarize this as $x = De^{t^2/2}$, $D \in \mathbf{R}$.

(c) $x = 1$ and $x = -1$ are constant solutions. All other solutions satisfy $x^2 - 1 \neq 0$, so they are found from $\frac{\dot{x}}{x^2-1} = 1 \iff t+C = \int\frac{dx}{x^2-1} = \frac{1}{2}\int\big(\frac{1}{x-1} - \frac{1}{x+1}\big)dx = \frac{1}{2}\big(\ln|x-1| - \ln|x+1|\big) = \frac{1}{2}\ln\big|\frac{x-1}{x+1}\big| \iff \frac{x-1}{x+1} = \pm e^{2(t+C)} = De^{2t}$, where $D = \pm e^{2t} \neq 0$. Now we can solve for x as follows, to obtain the nonconstant solutions: $\frac{x-1}{x+1} = \pm e^{2(t+C)} = De^{2t} \iff x - 1 = (x+1)De^{2t} \iff x(1 - De^{2t}) = 1 + De^{2t} \iff x = \frac{1+De^{2t}}{1-De^{2t}}$, where still $D \neq 0$. Letting $D = 0$ in this formula recovers the constant solution $x = 1$, which allows us to simplify the answer a little, but we still need to give the other constant solution $x = -1$ separately (it doesn't correspond to any $D \in \mathbf{R}$, but rather to the limiting case $D \to \infty$).

Answer: $x = -1$ or $x = \frac{1+De^{2t}}{1-De^{2t}}$, $D \in \mathbf{R}$.

Remark: Actually this answer isn't quite complete, since it doesn't specify the *interval of existence* for the solution $x(t)$, i.e., the maximal interval on the t-axis where $x(t)$ makes sense. Let's investigate how this interval depends on what initial value $x(t_0) = x_0$ that we impose! The initial values $x_0 = \pm 1$ correspond to the constant solutions $x(t) = \pm 1$, which of course exist for all $t \in \mathbf{R}$. For initial values $x_0 \neq \pm 1$, we can most easily determine the parameter D

from the equation $\frac{x-1}{x+1} = De^{2t}$ which appeared as an intermediate step in our solution above; this gives $D = \frac{x_0-1}{x_0+1} e^{-2t_0} \neq 0$. If $-1 < x_0 < 1$, then $D < 0$, so that the denominator $1 - De^{2t}$ in $x(t)$ stays positive always, implying that $x(t)$ exists for all $t \in \mathbf{R}$. (Note that the solution stays between ± 1 for all t, is decreasing, and satisfies $x(t) \to \mp 1$ as $t \to \pm\infty$.) But if $x_0 > 1$, then $0 < D < e^{-2t_0}$, so that the denominator becomes zero when $e^{2t} = 1/D > e^{2t_0}$, i.e. at some time $t = t^*$ such that $t^* > t_0$. (We could compute t^* in terms of x_0 if we like, but never mind.) This means that $x(t)$ "blows up in finite time": $x(t) \nearrow \infty$ as $t \nearrow t^*$. The formula for $x(t)$ is undefined only for $t = t^*$, but it's not really meaningful[2] to consider the values $x(t)$ obtained from that formula for $t > t^*$ as providing a solution to our initial value problem, since that part of the graph $x = x(t)$ is "disconnected" from the given initial value $x(t_0) = x_0$ by a vertical asymptote $t = t^*$. So the interval of existence for the solution $x(t)$ in the case $x_0 > 1$ is $(-\infty, t^*)$. Similarly, if $x_0 < -1$, then $e^{2t} < D$ and the denominator becomes zero at some time $t = t_*$ such that $t_* < t_0$. Now the solution "blows up in finite backwards time": $x(t) \searrow -\infty$ as $t \searrow t_*$. Thus, the interval of existence for the solution $x(t)$ in this case is (t_*, ∞).

(d) Note that the ODE can be written as $t^2\dot{x} = (x+1)^2$, with $x = -1$ as a constant solution. But since $x(-1)$ is not supposed to be -1, that's not the solution that we seek, so we look among the other solutions, given by $\frac{1}{(x+1)^2}\dot{x} = \frac{1}{t^2} \iff \frac{-1}{x+1} = \frac{-1}{t} + C$. Inserting $t = -1$ and $x = 1$ here shows that $C = -\frac{3}{2}$. Now solving for x gives the answer:

$$x(t) = \frac{t}{1 - Ct} - 1 = -\frac{t+2}{3t+2}.$$

The interval of existence is the largest interval on the t-axis which contains the point $t_0 = -1$ where the initial condition was given, but not the singular point $t^* = -\frac{2}{3}$ where the solution blows up. In other words, it's $(-\infty, -\frac{2}{3})$.

Remark: Another option is to use definite integrals:

$$\begin{aligned}
\int_{x(-1)}^{x(t)} \frac{dx}{(x+1)^2} = \int_{-1}^{t} \frac{d\tau}{\tau^2} \quad &\iff \quad \left[\frac{-1}{x+1}\right]_1^{x(t)} = \left[\frac{-1}{\tau}\right]_{-1}^{t} \\
&\iff \quad \frac{-1}{x(t)+1} - \frac{-1}{2} = \frac{-1}{t} - \frac{-1}{-1} \\
&\iff \quad x(t) = -\frac{t+2}{3t+2}.
\end{aligned}$$

(e) This is the same ODE as in the previous part, but now the initial condition $x(-1) = -1$ immediately implies that the solution which we seek is the constant solution $x(t) = -1$ (with $\mathbf{R}$ as the interval of existence)!

A3 (a–e) See problem A1.

(f) $x(t) = Ae^{-2t} + Be^{-4t} + (4t-3)/32 + te^{-2t}$.

(g) $x(t) = (At + B)e^{-3t}$.

(h) $x(t) = e^{-3t}(A\cos t + B\sin t) + te^{-3t}\sin t$.

(i) $x(t) = Ae^t + e^{-t/2}\big(B\cos\frac{\sqrt{3}x}{2} + C\sin\frac{\sqrt{3}x}{2}\big) + \frac{1}{3}xe^x + \frac{1}{2}(\cos x - \sin x)$.

A4 (a) The constant solutions are $x = 0$ and $x = K$. The non-constant solutions are given by

$$\int \frac{dx}{x\big(1 - x/K\big)} = \int r\,dt \quad \iff \quad rt + C = \int \left(\frac{1}{x} + \frac{1}{K-x}\right) dx = \ln|x| - \ln|K-x| = \ln\left|\frac{x}{K-x}\right|,$$

which gives $\frac{x}{K-x} = \pm e^{rt+C} = De^{rt}$, where $D = \pm e^C \neq 0$. Solving for x gives $x = \frac{KDe^{rt}}{1+De^{rt}}$. Here $D = 0$ gives the constant solution $x = 0$, so we now have that $x = K$ or $x = \frac{KDe^{rt}}{1+De^{rt}}$ with $D \in \mathbf{R}$.

[2] Unless we treat t as a *complex* variable, which we will not do in this course.

For $x(0) = x_0 \neq K$, we have $D = \frac{x_0}{K-x_0}$, which gives

$$x(t) = \frac{KDe^{rt}}{1+De^{rt}} = \frac{K\frac{x_0}{K-x_0}e^{rt}}{1+\frac{x_0}{K-x_0}e^{rt}} = \frac{Kx_0e^{rt}}{(K-x_0)+x_0e^{rt}} = \frac{Kx_0e^{rt}}{K+(e^{rt}-1)x_0},$$

as desired. And, as can be verified directly, this formula also happens to correctly give the constant solution $x(t) = K$ in the case $x_0 = K$, so it works for all $x_0 \in \mathbf{R}$!

(b) By writing the ODE as $\dot{x} - rx = -\frac{r}{K}x^2$ we see that it's a Bernoulli equation ("LHS is linear, RHS is a power of x"). We first note that there's a constant solution $x = 0$, and then we seek the other solutions by dividing by x^2 and letting $y(t) = 1/x(t)$:

$$\frac{\dot{x}-rx}{x^2} = -\frac{r}{K} \quad\Longleftrightarrow\quad -\frac{\dot{x}}{x^2} + \frac{r}{x} = \frac{r}{K} \quad\Longleftrightarrow\quad \frac{d}{dt}\left(\frac{1}{x}\right) + r\cdot\frac{1}{x} = \frac{r}{K} \quad\Longleftrightarrow\quad \dot{y} + ry = \frac{r}{K}.$$

This linear equation for y is easily solved (using an integrating factor, or $y(t) = y_{\text{hom}}(t) + y_{\text{part}}(t)$ with the very simple ansatz $y_{\text{part}}(t) = A$), which gives

$$y(t) = Ce^{-rt} + \frac{1}{K}.$$

An initial condition $x(0) = x_0 \neq 0$ corresponds to $y(0) = 1/x_0$, which gives $C = \frac{1}{x_0} - \frac{1}{K}$, so that

$$y(t) = \left(\frac{1}{x_0} - \frac{1}{K}\right)e^{-rt} + \frac{1}{K} = \frac{(K-x_0)e^{-rt} + x_0}{x_0K}.$$

And the reciprocal of this gives us the desired formula for $x = 1/y$,

$$x(t) = \frac{x_0K}{(K-x_0)e^{-rt} + x_0}.$$

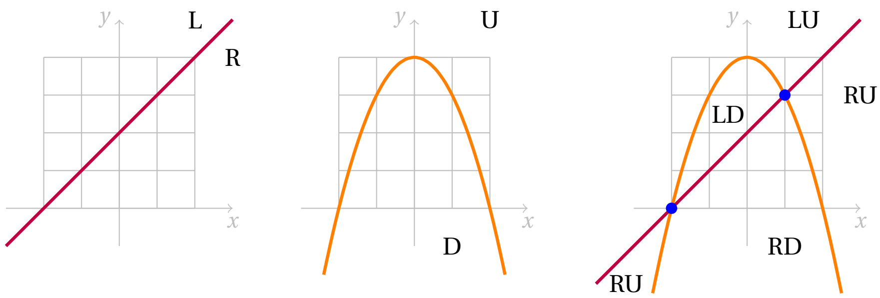
We derived this under the assumption $x_0 \neq 0$, but it's clear that the formula works for all $x_0 \in \mathbf{R}$, since it correctly gives the constant solution $x(t) = 0$ when $x_0 = 0$.

A5 $x(t) = 1 + \dfrac{t^3}{2\cdot 3} + \dfrac{t^6}{2\cdot 3\cdot 5\cdot 6} + \dfrac{t^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \dots$

A6 The paradox is the following: Accordingto the phase portrait, $x(t)$ should clearly tend to $-\infty$ if $x_0 < 0$, but the solution formula gives $\lim_{t\to\infty} x(t) = K$ for any $x_0 \neq 0$. A hint, if you can't figure out what's going on here, is that the logistic equation is rather similar to the ODE in problem A2c, and that the discussion in the solution to that problem might put you on the right track.

A7 $\dot{x} = (x^2 - t^2)/2tx$.

A9 (a) The equilibrium points are $(x, y) = (-2, 0)$ and $(1, 3)$.



A14 For example, $(u, v, w) = (x, y - xz, z)$ works.

(This shows that the nonlinear system in indeed locally topologically equivalent to its linearization at the hyperbolic equilibrium point $(0, 0, 0)$, as the linearization theorem promises. In this case, we actually happen to get more than that: the systems are even globally C^∞-conjugate, since the mapping $h(x, y, z) = (x, y - xz, z)$ is a bijection of class C^∞ from $\mathbf{R}^3$ to $\mathbf{R}^3$ with inverse of class C^∞.)

A17 With $V(x,y) = x^2 + cy^k$ we have

$$\begin{aligned}\dot V &= V'_x \dot x + V'_y \dot y \\ &= 2x(-x + 6y^3 - 3y^4) + cky^{k-1}(-x - y + \tfrac{1}{2}xy) \\ &= -2x^2 + 12xy^3 - 6xy^4 - ckxy^{k-1} - cky^k + \tfrac{1}{2}ckxy^k \\ &= (-2x^2 - cky^k) + x(12y^3 - cky^{k-1}) - \tfrac{1}{2}xy(12y^3 - cky^{k-1}).\end{aligned}$$

Taking $k = 4$ and $c = 3$, we get a positive definite function $V = x^2 + 3y^4$ such that $\dot V = -2x^2 - 12y^4$ is negative definite. So V is a strong Liapunov function for the system, and hence the origin is asymptotically stable. Moreover, it's *globally* asymptotically stable, since we also have $V(x,y) \to \infty$ as $\sqrt{x^2+y^2} \to \infty$.

A18 We have $V(x,y) = x^2(2-x^2) + y^4 > 0$ if $|x| < \sqrt{2}$ and $(x,y) \neq (0,0)$, so V is positive definite in the strip defined by the inequality $|x| < \sqrt{2}$. Furthermore,

$$\dot V = 4(x - x^3)\dot x + 2y\dot y = \big(4x^2(1-x^2) + 2y^2\big)(x^2 + y^2 - 1) < 0$$

for $0 < x^2 + y^2 < 1$, so $\dot V$ is negative definite in the open unit circle $\Omega = \{x^2 + y^2 < 1\}$, which is contained in the strip. So V is a strong Liapunov function in the open unit circle. Liapunov's theorem thus shows that the origin is asymptotically stable.

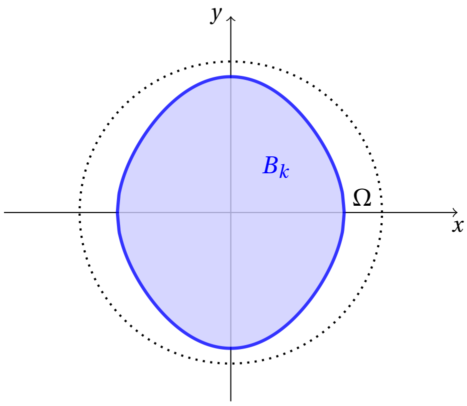
Next, for any $k \in (0,1)$, the sublevel set

$$B_k = \{(x,y) \in \Omega : V(x,y) \le k\}$$

is given by

$$B_k = \left\{(x,y) \in \mathbf{R}^2 : |x| \le \sqrt{1 - \sqrt{m}} \text{ and } |y| \le \sqrt{(1-x^2)^2 - m}\right\}, \quad \text{where } m = 1 - k.$$

(You may have to do a bit of careful thinking to convince yourself that this is true!) So that set is a closed topological ball contained in Ω:

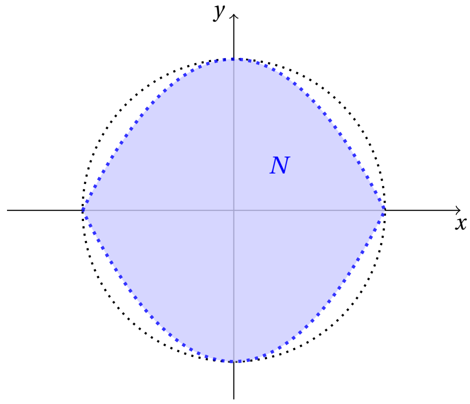


The smallest value α of $V(x,y)$ on the boundary ∂B is of course $\alpha = k$, since that's the *only* value of $V(x,y)$ on the boundary, so (according to the usual recipe) the set

$$N_k = \{(x,y) \in B_k : V(x,y) < \alpha\} = \operatorname{interior}(B_k)$$

is a domain of stability. Since this is true for any $k \in (0,1)$, we can get a larger domain by taking the union over all these N_k, giving the answer

$$N = \bigcup_{0<k<1} N_k = \{(x,y) \in \Omega : V(x,y) < 1\} = \left\{(x,y) \in \mathbf{R}^2 : |x| \le 1 \text{ and } |y| \le \left|1 - x^2\right|\right\}.$$



Remark: Note that the level set $V = 1$ is the union of the two parabolas $y = \pm(1-x^2)$, since $V(x,y) - 1 = (2x^2 + y^2 - x^4) - 1 = y^2 - (1-x^2)^2 = \big(y + (1-x^2)\big)\big(y - (1-x^2)\big)$, so that $V(x,y) - 1 = 0$ iff $y = \pm(1-x^2)$.

A19 We have $V(0,0) = 0$ and $V(x,y) = x^2(1 + \frac{1}{3}x) + y^2 > 0$ if $(x,y) \neq (0,0)$ and $x > -3$, and moreover

$$\dot V = (2x + x^2)\dot x + 2y\dot y = -2y(2x + x^2) + 2y(2x + x^2 - y^3) = -2y^4 \le 0$$

for all (x,y), so that V is a weak Liapunov function in the region $\Omega_1 = \{(x,y) : x > -3\}$. Hence, by Liapunov's theorem, the origin is a **stable** equilibrium. Next, consider the subregion $\Omega_2 = \{(x,y) : x > -2\}$, where V is obviously still a weak Liapunov function. The set of points in Ω where $\dot V = 0$, call it S, is the portion of the x-axis where $x > -2$. For $(0,0) \neq (x,y) \in S$, we have $\dot x = 0$ and $\dot y = 2x + x^2 = x(x+2) \neq 0$, so that the vector field is transversal to S except at the origin; hence the only complete trajectory contained in S is the origin, so the hypotheses for LaSalle's theorem are fulfilled, showing that the origin is in fact **asymptotically stable**. But it is **not globally asymptotically stable**, for the simple reason that there is another equilibrium point $(-2,0)$.

A21 $V(x,y) = 19x^2 + 8xy + 3y^2$.

A22 Hints: One can write $\dot V$ as

$$\dot V = -2\,\frac{x^2(1 - x^2y^2)^2 + y^2(1 + 2x^2)(1 + x^2y^2)}{(1 + x^2)^2},$$

and $\big(x(t), y(t)\big) = (e^{2t}, e^{-2t})$ is a particular solution of the system.

A23 $\dot r = r - r^3$, $\dot\theta = r^2(1 - \cos\theta)$. You can use the computer to check your phase portrait (Wolfram Alpha link).

A25 Allow me to write (x,y) instead of (x_1,x_2) for simplicity, so that the system reads

$$\dot x = y^2 - x^2, \qquad \dot y = 2xy.$$

Let's try to obtain the flow lines on the form $x = x(y)$, by solving the ODE

$$x'(y) = \frac{dx}{dy} = \frac{\dot x}{\dot y} = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right).$$

Introducing a new dependent variable $z(y)$ via the relation $x(y) = yz(y)$, we get $x'(y) = z(y) + yz'(y)$, so that

$$x' = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right) \quad\Longleftrightarrow\quad yz' + z = \frac{1}{2}\left(\frac{1}{z} - z\right) \quad\Longleftrightarrow\quad z' + \frac{3}{2y}\cdot z = \frac{1}{2y}\cdot\frac{1}{z}.$$

(Let's assume $y \neq 0$, since we're dividing by y.) This is now a Bernoulli equation, since the LHS in linear and the RHS is proportional to a power of z. So we divide by this power z^{-1}, i.e., we multiply by z, and get a linear ODE for the function $w(y) = z(y)^2$, which we can solve using the integrating factor $e^{3\ln y} = y^3$:

$$\cdots \iff zz' + \frac{3}{2y}\cdot z^2 = \frac{1}{2y} \iff w' + \frac{3}{y}\cdot w = \frac{1}{y} \iff (y^3 w)' = \frac{1}{y}\cdot y^3$$
$$\iff y^3 w = \tfrac{1}{3}y^3 + C \iff w = \frac{1}{3} + \frac{C}{y^3} \iff x(y) = yz(y) = \pm y\sqrt{w(y)} = \pm y\sqrt{\frac{1}{3} + \frac{C}{y^3}}.$$

We can solve for C to obtain a constant of motion $F(x,y)$ if we like,

$$C = y^3 w - \tfrac{1}{3}y^3 = y^3 z^2 - \tfrac{1}{3}y^3 = y^3(x/y)^2 - \tfrac{1}{3}y^3 = x^2 y - \tfrac{1}{3}y^3 =: F(x,y),$$

but the explicit form $x = x(y)$ above is more useful when drawing the phase portrait.

For $C = 0$ we simply have the lines $x = \pm y/\sqrt{3}$, and the curves $x = x(y)$ for $C \neq 0$ can be drawn using standard calculus methods. In addition, we can make use of the nullclines of the original system (i.e., the coordinate axes and the lines $y = \pm x$), as well as taking the signs of $\dot{x}$ and $\dot{y}$ into account (of course!). Then we also see what happens when $y = 0$, the case that we discarded above: along the line $y = 0$ we have solution curves "$\longleftarrow 0 \longleftarrow$". (This, by the way, is the simplest way of seeing that the origin is unstable in problem 3.22.)

You can check your phase portrait using the computer (Wolfram Alpha link).

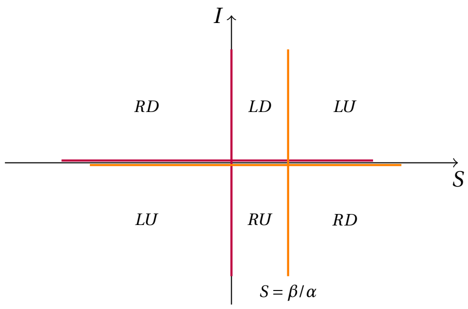
A27 Hint: To show that the solution curves spiral outwards, consider in what direction they deviate from the closed solution curves of the conservative system $\dot{x} = -y$, $\dot{y} = x(1 - y^2)$.

A28 Hints: What you need to check is that at every point (x,y) on the parabola $y = x^2$, the vector $(\dot{x}, \dot{y}) = (x^2 - x - y, x^2 - 3y)$ is a tangent vector to the parabola. Once this is done, you know that solutions starting on the parabola have to stay on it. Then letting $y = x^2$ in the equation for $\dot{y}$ gives $\dot{y} = -2y$; what does this tell you about the lifetime of the solution?

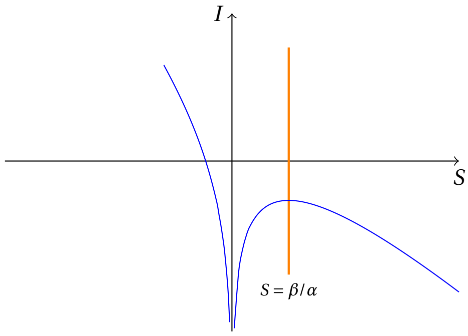
A30 (a) This is trivial, just add up the right-hand sides of the ODEs!

(b) Since $R \geq 0$, the relation $R = 1 - I - S$ means that $1 - I - S \geq 0$, or in other words $I + S \leq 1$.

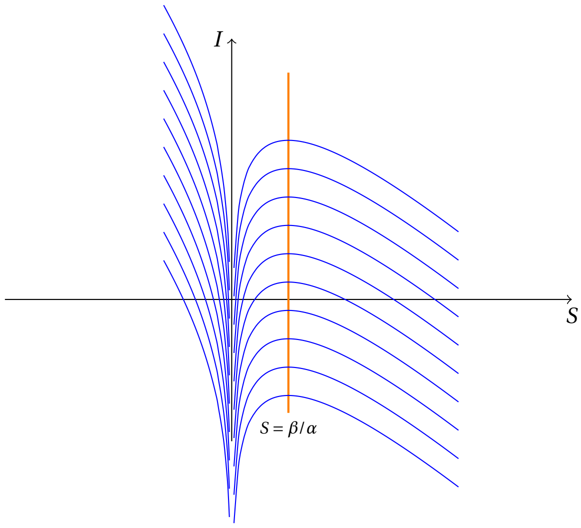
(c) *Every* point on the S-axis is an equilibrium point. (Which is natural – after all, if $I = 0$ then there is no infection spreading in the population, so everything will just remain as it is!)



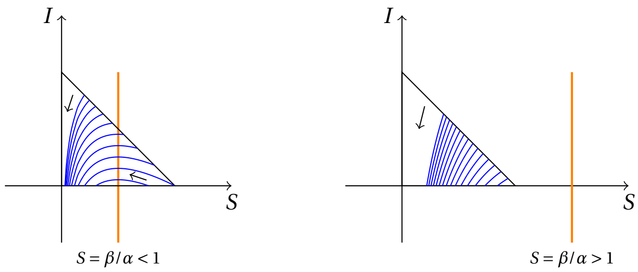
(d) For $IS \neq 0$ we get $dI/dS = (\alpha IS - \beta I)/(-\alpha IS) = -1 + \frac{\beta}{\alpha S}$, which immediately integrates to $I(S) = -S + \frac{\beta}{\alpha}\ln|S| + C$. So all these curves, for different values of C, are just translates in the I-direction of the curve for $C = 0$, $I(S) = -S + \frac{\beta}{\alpha}\ln|S|$. And that curve is fairly easy to draw using standard calculus methods:



The local maximum is $I\left(\frac{\beta}{\alpha}\right) = \frac{\beta}{\alpha}\left(-1+\ln\frac{\beta}{\alpha}\right)$, whose height depends on the value of β/α that we have chosen when drawing the graph. But this detail is quite irrelevant, since we are looking at the whole *family* of parallel curves anyway:



(e) The two cases, drawn in the triangle D only:



(f) As $t \to \infty$, the curve $(S(t), I(t))$ approaches some equilibrium point $(S^*, 0)$ on the positive S-axis, so the epidemic dies out, and there will be a fraction $S^* \in (0,1)$ of the population that never got infected, while the rest of the population (the fraction $R^* = 1 - S^*$) have had the disease and recoved. The value of S^* depends on the initial conditions (and on the value of β/α, of course).

If $\beta/\alpha < 1$, then the parameter β which measures the recovery rate is small in comparison to the parameter α which measures how contagious the disease is. In this case, a small number of infectives introduced into a susceptible population will lead to an outbreak, where $I(t)$ peaks when $S = \beta/\alpha$, after which the epidemic subsides (because more people are become immune, so that it's getting less and less likely for an infected individual to meet someone who is susceptible). But in the other case $\beta/\alpha > 1$, the recovery rate β is high in comparison with the contagiousness α – high enough that the number of infected will start decreasing right away.

A34 (a) The conditions for equilibrium are

$$\left(\frac{\beta_1 c}{\beta_2 + c} - 1\right) x = 0, \qquad 1 - c - \frac{\beta_1 c x}{\beta_2 + c} = 0.$$

The first condition holds iff $x = 0$ or $\frac{\beta_1 c}{\beta_2+c} = 1$. In the first case, $x = 0$, we get $c = 1$ from the second condition. The other case is what leads to the nontrivial equilibrium $(x, c) = (x^*, c^*)$; the condition $\frac{\beta_1 c^*}{\beta_2+c^*} = 1$ can be solved for c^* provided that $\beta_1 \neq 1$, yielding $c^* = \frac{\beta_2}{\beta_1-1}$, and it also turns the second condition into $1 - c^* - 1 \cdot x^* = 0$, so that $x^* = 1 - c^*$. Thus, the equilibria are

$$(x, c) = (0, 1), \qquad (x, c) = (x^*, c^*) = \left(1 - \frac{\beta_2}{\beta_1 - 1}, \frac{\beta_2}{\beta_1 - 1}\right).$$

Note that if $\beta_1 = 1 + \beta_2$, then $(x^*, c^*) = (0, 1)$, so in order to have two distict equilibrium points, we need to impose the condition $\beta_1 \neq 1 + \beta_2$ (in addition to $\beta_1 \neq 1$).

(b) In the process of finding (x^*, c^*) we found that $x^* = 1 - c^*$, so clearly (x^*, c^*) satisfies the line's equation $x + c = 1$. (It may be noted that the other equilibrium $(x, c) = (0, 1)$ also lies on that line.)

(c) The point (x^*, c^*) lies on the line $x + c = 1$, so if it's going to be in the positive quadrant, it must lie on the line segment between the points $(1, 0)$ and $(0, 1)$. This happens iff $0 < c^* < 1$, i.e.,

$$0 < \frac{\beta_2}{\beta_1 - 1} < 1.$$

The parameters β_1 and β_2 are both positive by definition, so the left inequality holds iff $\beta_1 - 1 > 0$. Under this condition, we can multiply the right inequality by the positive number $\beta_1 - 1$, to obtain $\beta_2 < \beta_1 - 1$. (Remember to carefully keep track of signs when working with inequalities! After all, if you multiply an inequality by a *negative* number, then it must be *reversed*.) So the double inequality holds iff $\beta_1 > 1$ and $\beta_1 > 1 + \beta_2$, where the first condition can be removed, since it's automatically fulfilled if the second one is. So in the end, we're left with just the condition $\beta_1 > 1 + \beta_2$.

(d) If $0 < \beta_1 < 1$, then $\beta_1 - 1 < 0$, so that $c^* = \frac{\beta_2}{\beta_1-1} < 0$, which means that (x^*, c^*) is on the part of the line $x + c = 1$ that lies in the fourth quadrant (if we draw the x-axis horizontally and the c-axis vertically).

If $1 < \beta_1 < 1 + \beta_2$, then $0 < \beta_1 - 1 < \beta_2$, so $c^* = \frac{\beta_2}{\beta_1-1} > 1$, and (x^*, c^*) is on the part of the line $x + c = 1$ that lies in the second quadrant.

(e) Computing the derivatives, we find

$$J(x, c) = \begin{pmatrix} \frac{\beta_1 c}{\beta_2+c} - 1 & \frac{\beta_1\beta_2 x}{(\beta_2+c)^2} \\ -\frac{\beta_1 c}{\beta_2+c} & -1 - \frac{\beta_1\beta_2 x}{(\beta_2+c)^2} \end{pmatrix},$$

so

$$J(0, 1) = \begin{pmatrix} \frac{\beta_1}{\beta_2+1} - 1 & 0 \\ -\frac{\beta_1}{\beta_2+1} & -1 \end{pmatrix}$$

and

$$J(x^*, c^*) = \begin{pmatrix} \frac{\beta_1 c^*}{\beta_2+c^*} - 1 & \frac{\beta_1\beta_2 x^*}{(\beta_2+c^*)^2} \\ -\frac{\beta_1 c^*}{\beta_2+c^*} & -1 - \frac{\beta_1\beta_2 x^*}{(\beta_2+c^*)^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{(1-\beta_1)(\beta_2+1-\beta_1)}{\beta_1\beta_2} \\ -1 & -1 - \frac{(1-\beta_1)(\beta_2+1-\beta_1)}{\beta_1\beta_2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{(1-\beta_1)(\beta_2+1-\beta_1)}{\beta_1\beta_2} \\ -1 & -\frac{(1-\beta_1)^2+\beta_2}{\beta_1\beta_2} \end{pmatrix}.$$

A36 (a) Hint: The perpendicular distance from the point $\big(x(\tau), c(\tau)\big)$ to the line $x + c = 1$ is

$$\left|\frac{x(\tau) + c(\tau) - 1}{\sqrt{2}}\right|.$$

Let $y(\tau) = x(\tau) + c(\tau) - 1$, compute $y'(\tau)$, show that it can be expressed solely in terms of $y(\tau)$, and solve that ODE for $y(\tau)$. What does the result tell you about the distance $|y(\tau)|/\sqrt{2}$?

A37 (a) We have two expressions for g, both of which are useful:

$$g(c) = \frac{(1-c)(\beta_2 + c)}{\beta_1 c} = -\frac{1}{\beta_1}c + \frac{1-\beta_2}{\beta_1} + \frac{\beta_2}{\beta_1 c},$$

In the second expression, the third term $\frac{\beta_2}{\beta_1 c}$ tends to zero as $c \to \pm\infty$. This means that the straight line given by the first two terms, $x = -\frac{1}{\beta_1}c + \frac{1-\beta_2}{\beta_1}$, is a skew asymptote to the curve $x = g(c)$. We also have $g(c) \to \pm\infty$ as $c \to 0^{\pm}$ because of the c in the denominator of the third term, and $g(c)$ is of course undefined at $c = 0$.

Being the c-nullcline, the curve of course passes through both equilibria $(0,1)$ and (x^*, c^*), and also through the point $(x, c) = (0, -\beta_2)$, as we see from the factor $\beta_2 + c$ in the first expression for g.

Taking derivatives (using the second expression), we find

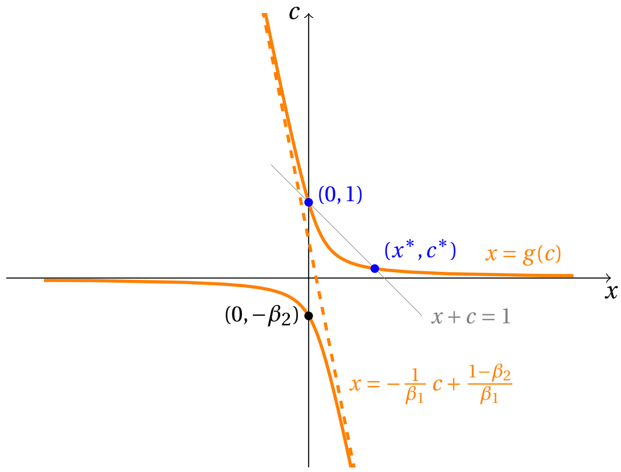
$$g'(c) = -\frac{1}{\beta_1} - \frac{\beta_2}{\beta_1 c^2} = -\frac{c^2 + \beta_2}{\beta_1 c^2} < 0 \quad \text{for all } c \neq 0,$$

which means that g is decreasing on the interval $c < 0$ and on the interval $c > 0$ (although it's *not* a decreasing function as a whole!), and

$$g''(c) = \frac{2\beta_2}{\beta_1 c^3},$$

which has the same sign as c, so that g is concave on the interval $c < 0$ and convex on the interval $c > 0$.

Now we have everything that we need in order to draw the graph. I'm drawing the x-axis horizontally, which means that the graph $x = g(c)$ will be "flipped" compared to the usual way of drawing graphs. I hope you will not find this too confusing! The graph is drawn with $\beta_1 = 5$ and $\beta_2 = 1/2$ (so that $\beta_1 > 1 + \beta_2$, the good case).



Note that the argument about convexity is necessary in order to show that the curve really lies at it is drawn in relation to the line $x + c = 1$. It's *not* enough to just check that g is decreasing – it could be decreasing and pass through the equilibria, yet still stay *above* the line, right?

A40 (a) Exact solution $x(t) = e^t$. Picard iterates $x_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!}$.

A44 (b) Hint: Assume, for a contradiction, that some solution $\mathbf{x}(t)$ only exists for $0 \le t < t_0$. From

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{X}(\mathbf{x}(s))\,ds,$$

deduce that

$$|\mathbf{x}(t)| \le |\mathbf{x}_0| + at + \int_0^t b|\mathbf{x}(s)|\,ds \qquad \text{for } 0 \le t < t_0,$$

and apply Grönwall's lemma from problem A43 to show that $|\mathbf{x}(t)|$ cannot tend to ∞ as $t \nearrow t_0$.

A45 $\alpha^2 = 1/4$, $x(t) = t^{-1/2}\cos t$.

A47 (a) $x(t) = (At+B)e^{-t} - \cos t$.

(b) $x(t) = A\cos 3t + B\sin 3t + \frac{1}{6}t\sin 3t$.

(c) $x_1(t) = A(\cos t + 2\sin t) + B(-5\sin t) + 8t - 1$,
$x_2(t) = A\sin t + B(\cos t - 2\sin t) + 4t - 2$.

(d) $x(t) = A\cos t + B\sin t + \frac{1}{2}\cos t \cdot \ln\left|\frac{1-\sin t}{1+\sin t}\right|$.

(e) $x(t) = At + B(1+t^2) - t^2 + t\ln\left|\frac{1+t}{1-t}\right| + \frac{1}{2}(1+t^2)\ln\left|1-t^2\right|$.

A48

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t}\arctan t \begin{pmatrix} 1-2t \\ -t \end{pmatrix} + e^{2t}\ln(1+t^2)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + e^{2t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

A49 Hints:

(a) Consider the Wronskian $W = y\dot{z} - \dot{y}z$ of $y(t) = t$ and $z(t) = \cos t$.

(b) Just plug $x(t) = t$ and $x(t) = \cos t$ into the ODE and see what $p_0(t)$ and $p_1(t)$ must be in order for the ODE to be satisfied. In your answer, you should be able to see why there's a problem when the Wronskian becomes zero.

(c) Same as for (a).

[Table of contents]

All problems listed on one page

Lecture 1. A1, A2, 1.4, A3, A4, A5, 1.11, 1.12, 1.13, A6, 1.14, 1.17*, A7

Lecture 2. A8, A9, A10, A11, 1.20, 1.23, 1.25, 1.32, 1.36

Lecture 3. 2.1abd, 2.3a, 2.4, 2.8, 2.9

Lecture 4. 2.13, A12, 2.22, 2.23, 2.29abcd, 2.30, 2.33, 2.35

Lecture 5. 3.5, 3.6, 3.7, A13, 3.8, 3.11, A14*, A15**

Lecture 6. 3.13abe, 3.14abe, 3.15, A16, A17, A18*, A19, 3.19b, A20, 3.18*, A21*, A22, A23, 3.22, A24*, 3.24, 3.28ab, 3.29*, A25*

Lecture 7. 3.35, A26, A27*, 3.36, 3.42, 3.43, A28, A29

Lecture 8. 5.2, 5.3, 5.10, 5.15, A30, A31, A32, A33, A34, A35, A36, A37, A38, A39

Lecture 9. A40, A41, A42, A43, A44

Lecture 10. A45, A46, A47, A48, A49