Tekniska Högskolan i Linköping, Matematiska Institutionen
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Tentamen i Ordinära Differentialekvationer och dynamiska system 2011-12-15, kl. 8-13,
Allowed aids - a scientific calculator. **Kurskod: TATA71, NMAC 26**
Provkod: TEN 1

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find a general solution of the Bernoulli type first order ODE $y' = (-\frac{1}{1+x^2})y^2$

(a) By reducing this Bernoulli type equation to a linear first order ODE.
(b) By separating the variables in this equation (control the solution).
(c) Find a solution satisfying the initial condition $y(0) = 1$ and $y(0) = 0$.

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2. Find a general solution of the following system of equations

$$\dot{x} = 2x - y - 2, \qquad \dot{y} = 3x - 2y - 2$$

Draw an approximate phase portrait for this system.

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3. Use the method of reduction of order to find a second linearly independent solution for $(1-x^2)y'' + 2xy' - 2y = 0, \,(-1 < x < 1)$ when one solution $z(x) = x$ is known.
Show that it is linearly independent of the first solution.

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4. Consider the dynamical system in the plane: $\dot{x} = y - 1,\ \dot{y} = x^2 - y$

Find all equilibrium points and investigate their linear stability. Draw the vectorfield on the nullclines. What is the expected character of trajectories in the neighbourhood of each equilibrium point?

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5. Show, by finding a suitable Liapunov function, that the point $(x = 0, y = 0)$ is a stable equilibrium point of the dynamical system

$$\dot{x} = 3y - 4xy^2, \qquad \dot{y} = -2x + 2x^2 y$$

Then prove that $(0,0)$ is also asymptotically stable by referring to the relevant Liapunov theorem. What the linear criterion of stability says about stability of $(0,0)$?

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6. Formulate the theorem about existence and uniqueness of solutions in a form that you find suitable for showing that the initial value problem: $y'(x) = (\sin x)|y| + h(x),\ y(1) = 2$ with continuous $h(x)$ has a unique solution. Give an equivalent integral equation satisfied by this solution.

Solutions TATA71 2011-12-15

1. (a) The substitution $y(x) = 1/z(x)$, $z(x) \neq 0$ gives $y'(x) = -z'(x)/z(x)^2$ and $z' = \frac{1}{1+x^2}$.
So $z(x) = \arctan x + C$ and $y(x) = 1/(\arctan x + C)$.

(b) By separating variables we get $y'/y^2 = -\frac{1}{1+x^2}$ if $y(x) \neq 0$. So $-1/y = -\arctan x + C$ and again $y(x) = 1/(\arctan x + C)$. We also have to check whether the condition $y(x) \neq 0$ does not exclude a possible solution. Substitution into the equation shows that $y(x) = 0$ is also a solution.

(c) From the IC $1 = y(0) = \frac{1}{C}$ the solution is $y(x) = 1/(\arctan x + 1)$. The second IC $y(0) = 0$ is satisfied by $y(x) = 0$.

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2. From $2x - y - 2 = 0, 3x - 2y - 2 = 0$ the equilibrium point is $(x_0 = 2, y_0 = 2)$. The linear change of variables $x = u + 2$, $y = w + 2$ moves the equilibrium point to $(0,0)$ and (u, w) satisfies the homogeneous system $\dot{u} = 2u - w$, $\dot{w} = 3u - 2w$. $\det\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix} = 0$ gives eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ and the eigenvectors $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u(t)+2 \\ w(t)+2 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t + B\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.
For (u, w) - system the origin is an unstable equilibrium of the saddle point type. The line through the origin along the eigenvector w_1 contains two outgoing trajectories and the line along the eigenvector w_2 contains two ingoing trajectories. The remaining trajectories are hyperbolas approaching asymptotically these lines and they are directed consistently with the directions of trajectories along the lines w_1 and w_2.
The phase space diagram for the (x, y) system has the equilibrium point shifted to the point $(2,2)$.

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3. The ansatz for solution $y(x) = u(x)z(x) = xu(x)$ gives $(1 - x^2)xu'' + 2u' = 0$. The unknown $w = u'$ satisfies a first order separable ODE $(1 - x^2)xw' + 2w = 0$. So $w'/w = 2/(x^2 - 1)x$ and $w'/w = [-2/x] + [1/(x+1)] + [1/(x-1)]$. By integrating $w = \pm e^C(x^2 - 1)/x^2 = \pm e^C(1 - 1/x^2)$ and $u(x) = \int w(x)dx = \pm e^C(x + 1/x) + D = \pm e^C(x^2 + 1)/x + D$. From the second solution $y(x) = ux = e^C(x^2 + 1) + Dx$ one can take $y(x) = (x^2 + 1)$ as the simplest linearly independent solution. The determinant of the Wronskian matrix $Det\begin{bmatrix} z & y \\ z' & y' \end{bmatrix} = Det\begin{bmatrix} x & x^2+1 \\ 1 & 2x \end{bmatrix} = x^2 - 1 \neq 0$ for $-1 < x < 1$ proves linear independence.

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4. Equations $y - 1 = 0$, $x^2 - y = 0$ have two solutions $(x_1 = 1, y_1 = 1)$, $(x_2 = -1, y_2 = 1)$.
The Jacobian matrix is $J = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$. At the equilibrium points: $J_1 = J(x_1, y_1) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ has the eigenvalues $\lambda_+ = 1, \lambda_- = -2$ and the eigenvectors $\mathbf{w}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w}_- = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. $J_2 = J(x_2, y_2) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ has the eigenvalues $\lambda_\pm = -\frac{1}{2}(1 \pm i\sqrt{7})$ and no real eigenvectors. So $(1,1)$ is a saddle point and $(-1,1)$ is a spiral sink. The direction of the vectorfield on the nullclines: $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ x^2 - 1 \end{bmatrix}$ on $y - 1 = 0$,
$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} y-1 \\ 0 \end{bmatrix}$ on $x^2 - y = 0$ is consistent with this conclusion.

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5. $(0,0)$ is the only equilibrium point for the system $\dot{x} = 3y - 4xy^2$, $\dot{y} = -2x + 2x^2 y$. Try $V(x,y) = Ax^2 + By^2$, A, B-constant, $\dot{V}(x,y) = 2Ax\dot{x} + 2By\dot{y} = 2Ax(3y - 4xy^2) + 2By(-2x + 2x^2 y)$. $= (6A - 4B)xy + (4B - 8A)x^2 y^2$. Choose $A = 2, B = 3$ then the function $V(x,y) = 2x^2 + 3y^2$ is positive definite: $V(0,0) = 0$ and $\dot{V}(x,y) = -4x^2 y^2 \le 0$ has negative semidefinite derivative. The set $M = \{x : 0 = \dot{V}(x,y) = -12x^2 y^2\} = \{(x, y = 0)\} \cup \{(x = 0, y)\}$ consist of whole x-axis and y-axis. If a solution satisfies $y(t) = 0$ then $\dot{x} = 3y - 4xy^2 = 0$, $x(t) = const$ and by $0 = \dot{y} = -2x + 2x^2 y(t) = -2x$ $x(t) = const = 0$. Similarly if $x(t) = 0$ then also $y(t) = 0$. So only the constant solution $(x(t) = 0, y(t) = 0)$ belongs to M and by the Liapunov theorem the point $(0,0)$ is asymptotically stable.

The Jacobian $J_0 = J(0,0) = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}$ has eigenvalues $\lambda_\pm = \pm i\sqrt{6}$ so the linear criterion of stability does not excludes that $(0,0)$ is stable but does not give a conclusive answer either.

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6. The right hand side of the equation $y'(x) = (\sin x)|y| + h(x)$ is a continuous function of both arguments. It satisfies a the Lipshitz inequality $|(\sin x)|y| + h(x) - (\sin x)|z| + h(x)| = |(\sin x)|y| - (\sin x)|z|| =$ $|(\sin x)||y| - |z|| \le |(\sin x)||y - z| \le |y - z|$ with the Lipshitz constant $L = 1$. Thus assumptions of the existence and uniqueness theorem are satisfied and the initial value problem has a unique solution that is defined for all values of x. By integrating both sides of the equation we immediately get

$y(x) - y(1) = \int_1^x y'(s)ds = \int_1^x [(\sin s)|y(s)| + h(s)]ds$ so $y(x) = 2 + \int_1^x [(\sin s)|y(s)| + h(s)]ds$ is the required integral equation.