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Tentamen i Ordinära Differentialekvationer och Dynamiska System   2012-12-19, 14-19,
No aids allowed.   Kurskod: TATA71 (NMAC 26), Provkod: TEN 1

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. For a linear inhomogeneous ODE of first order $y' + 2xy = x\exp(-x^2)$,

a) Find a general solution of the homogeneous equation.
b) Find a general solution of the inhomogeneous equation.
c) Determine a particular solution of the inhomogeneous equation that satisfies the condition $y(0) = 0$.

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2. Find equilibrium points and decide their stability for the dynamical system $\begin{cases} \dot{x} = -3x + 2y + 1 \\ \dot{y} = -3x + 4y - 1 \end{cases}$.

Determine a solution of this system going through the point (0,0).

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3. Find through the Lagrange´ method a particular solution of the differential equation $y'' + 3y' + 2y = 2x$. Confirm your solution by taking a polynomial ansatz for $y_p(x)$.

Give a solution satisfying the initial value problem $y(0) = -\frac{3}{2}$, $y'(0) = 1$.

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4. A dynamical system is given by equations

$$\dot{x} = x^2 + y^2 - 9, \qquad \dot{y} = xy + \sqrt{5}y$$

(a) Find all equilibrium points and decide their linear stability.
(b) Draw nullclines and direction of the vectorfield on the nullclines.
(c) Show that the line $y = 0$ contains trajectories of this system. How many trajectories stay on this line?

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5. Show by finding a suitable Liapunov function of the form $V(x,y) = Ax^2 + By^2$ and by referring to the relevant Liapunov theorem that the equilibrium point $(0,0)$ is asymptotically stable for the dynamical system

$$\dot{x} = y - 3xy^2, \qquad \dot{y} = -4x + 5x^2 y$$

What does the linear criterion of stability say about stability of $(0,0)$?

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6. Find a continuously differentiable function $y(x)$ that solves the integral equation

$$y(x) + 1 - \sin x = -\int_0^x [y(s)/(s+3)]ds$$

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Solutions ODE´s, TATA71, 2012-12-19

1. The homogeneous equation $y'+2xy=0$ can be solved by separating variables. When $x \neq 0$ and $y \neq 0$ it gives $y'/y=-2x$. By integrating both sides w.r.t. x we get $\ln|y|=-x^2+C$ and $y=\pm e^C \exp(-x^2)$. Since $y(x)\equiv 0$ is also a solution the general solution of the homogeneous equation can be written as $y_{\text{hom}}=D\exp(-x^2)$ with $D\in\mathbf{R}$. A general solution of the inhomogeneous equation is easiest to determine through the integrating factor method. The integrating factor is $\exp(\int(2x)dx)=\exp(x^2)$ and by multiplying both sides we obtain $[y\exp(x^2)]'=\exp(x^2)[y'+2xy]=x$, $y\exp(x^2)=\frac{1}{2}x^2+D$ and $y_{gen}=\frac{1}{2}x^2\exp(-x^2)+D\exp(-x^2)$. We see that this general solution is a sum of the general solution of the homogeneous equation $y_{\text{hom}}=D\exp(-x^2)$ and of the particular solution $y_{part}=\frac{1}{2}x^2\exp(-x^2)$. By substituting it into the inhomogeneous equation we verify that it satisfies the equation. From $0=y(x=0)=C$ we see that $y_{part}=\frac{1}{2}x^2\exp(-x^2)$ satisfies the condition $y(0)=0$.

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2. Equations $-3x+2y+1=0, -3x+4y-1=0$ give the equilibrium point is $(x_0=1, y_0=1)$. The Jacobian matrix $J=\begin{bmatrix}-3 & 2\\ -3 & 4\end{bmatrix}$ is constant. From $0=Det(J-\lambda)=\begin{bmatrix}-3-\lambda & 2\\ -3 & 4-\lambda\end{bmatrix}=\lambda^2-\lambda-6=(\lambda+2)(\lambda-3)$ eigenvalues are $\lambda_+=3, \lambda_-=-2$ and the equilibrium (1,1) is unstable. A general solution is a sum of the general solution of the homogeneous system and of a particular solution of the inhomogeneous system. A constant ansatz for a particular solution gives a constant solution $(x_p=1, y_p=1)$ that describes the equilibrium point.

Eigenvectors corresponding to the eigenvalues $\lambda_+=3, \lambda_-=-2$ are $w_+=\begin{bmatrix}1\\3\end{bmatrix}$, $w_-=\begin{bmatrix}2\\1\end{bmatrix}$. The general solution is

$\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=A\begin{bmatrix}1\\3\end{bmatrix}e^{3t}+B\begin{bmatrix}2\\1\end{bmatrix}e^{-2t}+\begin{bmatrix}1\\1\end{bmatrix}$. From the initial condition $\begin{bmatrix}0\\0\end{bmatrix}=A\begin{bmatrix}1\\3\end{bmatrix}+B\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}$, $A=-\frac{1}{5}, B=-\frac{2}{5}$

And the solution is $\begin{bmatrix}x(t)\\y(t)\end{bmatrix}=-\frac{1}{5}\begin{bmatrix}1\\3\end{bmatrix}e^{3t}-\frac{2}{5}\begin{bmatrix}2\\1\end{bmatrix}e^{-2t}+\begin{bmatrix}1\\1\end{bmatrix}$.

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3. The homogeneous equation $0=y''+3y'+2y=[y(x)=e^{\lambda x}]=e^{\lambda x}(\lambda^2+3\lambda+2)=e^{\lambda x}(\lambda+1)(\lambda+2)$ gives two solutions $y_1(x)=e^{-x}, y_2(x)=e^{-2x}$ and the general solution of the homogeneous equation is $y_h(x)=Ae^{-x}+Be^{-2x}$. We seek $y_p(x)=u(x)y_1(x)+w(x)y_2(x)=u(x)e^{-x}+w(x)e^{-2x}$. From the inhomogeneous equation by using the auxiliary condition $u'(x)e^{-x}+w'(x)e^{-2x}=0$ we get $u'(x)y_1'(x)+w'(x)y_2'(x)=u'(x)(-e^{-x})+w'(x)(-2e^{-2x})=2x$. Together with the auxiliary condition $u'(x)e^{-x}+w'(x)e^{-2x}=0$, we get 2 algebraic equations for $u'(x), w'(x)$. This leads to the solution $y_p(x)=-y_1(x)\int y_2(x)\frac{h(x)}{W(x)}dx+y_2(x)\int y_1(x)\frac{h(x)}{W(x)}dx=x-\frac{3}{2}$ where the $W(x)$ is Wronskian function $W(x)=y_1(x)y_2'(x)-y_2(x)y_1'(x)=-e^{-3x}$. $y_p(x)=x-\frac{3}{2}$ satisfies the initial conditions $y(0)=-\frac{3}{2}$, $y'(0)=1$.

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4. (a) Equations $x^2+y^2-9=0$, $(x+\sqrt{5})y=0$ have four solutions $(x_1=3, y_1=0)$, $(x_2=-3, y_2=0)$,

$(x_3 = -\sqrt{5}, y_3 = 2)$, $(x_4 = -\sqrt{5}, y_4 = -2)$ .The Jacobian matrix is $J = \begin{bmatrix} 2x & 2y \\ y & x+\sqrt{5} \end{bmatrix}$. At the equilibrium points:

$J_1 = J(3,0) = \begin{bmatrix} 6 & 0 \\ 0 & 3+\sqrt{5} \end{bmatrix}$ has the eigenvalues $\lambda_+ = 6, \lambda_- = 3+\sqrt{5}$, $J_2 = J(-3,0) = \begin{bmatrix} -6 & 0 \\ 0 & -3+\sqrt{5} \end{bmatrix}$ has $\lambda_+ = -6, \lambda_- = -3+\sqrt{5}$, $J_3 = J(-\sqrt{5},2) = \begin{bmatrix} -2\sqrt{5} & 4 \\ 2 & 0 \end{bmatrix}$ has $\lambda_+ = -\sqrt{5}+\sqrt{13}, \lambda_- = -\sqrt{5}-\sqrt{13}$.

$J_4 = J(-\sqrt{5},-2) = \begin{bmatrix} -2\sqrt{5} & -4 \\ -2 & 0 \end{bmatrix}$ has $\lambda_+ = -\sqrt{5}+\sqrt{13}, \lambda_- = -\sqrt{5}-\sqrt{13}$. We see that only $(x_2 = -3, y_2 = 0)$ has both eigenvalues $\lambda_+ = -6 < 0, \lambda_- = -3+\sqrt{5} < 0$ negative so that it is a stable equilibrium. The 3 remaining equilibria have a positive eigenvalue and are unstable.

(b) Denote $f(x,y) = x^2 + y^2 - 9$, $g(x,y) = (x+\sqrt{2})y$. The nullcline $f(x,y) = x^2 + y^2 - 9 = 0$ is a circle of radius 3 and the nullclines $g(x,y) = (x+\sqrt{2})y = 0$ are the x-axis and the vertical line having $x = -\sqrt{2}$. All equilibrium points stay on the circle at intersections with these 2 lines. The vectorfield on the $f(x,y) = x^2 + y^2 - 9 = 0$ is vertical and on the nulclines $x = -\sqrt{2}, y = 0$ is horisontal.

(c) When $y = 0$ then the second equation $\dot{y} = xy + \sqrt{5}y$ vanishes and te first one simplifies to $\dot{x} = x^2 - 9$. This means that the condition $y = 0$ is preserved by the equations. On the nullcline $y = 0$ the vectorfield is parallel to the x-axis. This axis consists of 5 trajectories: two equilibrium points $(x_1 = 3, y_1 = 0)$, $(x_2 = -3, y_2 = 0)$, the open interval $]-3,3[$ and the halvlines $]-\infty,-3[$, $]3,+\infty[$.

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5. From $y - 3xy^2 = 0$, $-4x + 5x^2y = 0$ only $(0,0)$ is an equilibrium point. Try $V(x,y) = Ax^2 + By^2$, A, B- constant, $\dot{V}(x,y) = 2Ax\dot{x} + 2By\dot{y} = 2Ax(y - 3xy^2) + 2By(-4x + 5x^2y) = (2A - 8B)xy + (10B - 6A)x^2y^2$. Choose $A = 4, B = 1$ then the function $V(x,y) = 4x^2 + y^2$ is positive definite: $V(0,0) = 0$, $V(x,y) > 0$, for $(x,y) \neq (0,0)$. $\dot{V}(x,y) = -14x^2y^2 \leq 0$ has negative semidefinite derivative. This is a sufficient condition for Liapunov stability of the point $(0,0)$.

The set $M = \{x : 0 = \dot{V}(x,y) = -12x^2y^2\} = \{(x, y = 0)\} \cup \{(x = 0, y)\}$ consist of whole x-axis and of whole y-axis. If a solution satisfies $y(t) = 0$ then $\dot{x} = y - 3xy^2 = 0$, $x(t) = const$ and by $0 = \dot{y} = -4x + 5x^2y(t) = -4x$ we have $x(t) = const = 0$. Similarly if $x(t) = 0$ then also $y(t) = 0$. So only the constant solution $(x(t) = 0, y(t) = 0)$ belongs to M and by the Liapunov theorem the point $(0,0)$ is also asymptotically stable.

The Jacobian $J(0,0) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ has eigenvalues $\lambda_\pm = \pm 2i$ so the linear criterion of stability does not excludes that $(0,0)$ is stable but does not give a conclusive answer either.

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6. By taking derivative of the integral equation we get $y'(x) + \frac{1}{x+3}y(x) = \cos x$, which is a first order linear ODE . By setting $x = 0$ we read the initial condition $y(0) = -1$. It has an integrating factor $(x+3)$. So $[(x+3)y]' = (x+3)\cos x$ and by integrating $(x+3)y = \int (x+3)\cos x dx = x\sin x + \cos x + 3\sin x + C$. From the initial condition $-1(3) = 1 + C$ and $C = -4$. Thus the solution is $y = \sin x + \frac{1}{x+3}[\cos x - 4]$.