Tekniska Högskolan i Linköping, Matematiska Institutionen
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Tentamen i Ordinära Differentialekvationer och Dynamiska System 2013-03-25 kl.14 - 19
No aids allowed. Kurskod: TATA71 (NMAC 26), Provkod: TEN

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find through the integrating factor method a general solution for the first order differential equation

$$y' + 2xy = 2x$$

Confirm the result by solving this equation also through the method of separation of variables. Give a solution satisfying the initial condition $y(0) = 1$.

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2. Which equilibrium points of the autonomous dynamical system $\dot{y} = y^3 - 2y^2 - y + 3$ are stable.

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3. Find a general solution of the following system of equations: $\dot{x} = \frac{1}{2}x + \frac{3}{2}y,\quad \dot{y} = \frac{3}{2}x + \frac{1}{2}y$
and draw the corresponding phase space picture. What can you say about stability of origin (0,0) ?

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4. A linear, inhomogeneous 2nd order differential equation $y'' + p(x)y' + q(x)y = h(x)$ has three particular solutions $y_1(x),\ y_2(x),\ y_3(x)$ that are linearly independent.
a) Give a formula for the general solution of this equation.
b) Express coefficients of the equation $p(x), q(x), h(x)$ through these particular solutions and their derivatives.

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5. A dynamical system is given by equations $\dot{x} = xy - 2x,\qquad \dot{y} = -xy + 6 - 2y$
Show that the system has 2 equilibrium points and decide their linear stability. Draw all nullclines and direction of the vectorfield on the nullclines. Show that if $x(t), y(t)$ solve the dynamical system then $\frac{d}{dt}(x + y - 3) = -2(x + y - 3)$. Show that the line $(x + y - 3) = 0$ contains 5 trajectories of this system.

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6. Find all equilibrium points for the dynamical system $\dot{x} = y,\quad \dot{y} = -x^3 + 4x$.
Show that trajectories of the dynamical system are level curves of the energy function $E(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - 2x^2$. Show that $E(x,y) + 4$ can be taken as a suitable Liapunov function and conclude that the equilibrium point $(x = 2, y = 0)$ is stable but not asymptotically stable.

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Solutions ODE´s (TATA71) 2013-03-25

1. An integrating factor is $\exp(\int(2x)dx) = \exp(x^2)$ and by multiplying with it both sides of the equation we obtain $[y\exp(x^2)]' = \exp(x^2)[y' + 2xy] = \exp(x^2) + C$, so $y = 1 + C\exp(-x^2), C \in \mathbf{R}$ and the condition $1 = y(1) = 1 + C\exp(0) = 1 + C$ gives $C = 0$. So $y(x) = 1 = const$ is the required solution.
For separating variables rewrite the equation as $y' = 2x(1 - y)$. When $x \neq 0$ and $y - 1 \neq 0$ it gives $y'/(y-1) = -2x$. By integrating both sides w.r.t. x we get $\ln|y - 1| = -x^2 + D,\ y - 1 = \pm e^D \exp(-x^2)$ and $y = 1 + C\exp(-x^2)$ where $C = \pm e^D \neq 0$ is an arbitrary real nonzero constant. Direct control shows that $y = 1 = const$ is also a solution. Thus we get again the same general solution $y = 1 + C\exp(-x^2), C \in \mathbf{R}$.

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2. Denote $f(y) = y^3 - 2y^2 - y + 2 = (y+1)(y-1)(y-2)$ then $f(y) = 0$ gives 3 equilibrium points $y_1 = -1, y_2 = 1, y_3 = 2$. $f'(y) = 3y^2 - 4y - 1$ and $f'(-1) = 6 > 0,\ f'(1) = -2 < 0,\ f'(2) = 3 > 0$. Only the point $y_2 = 1$ is stable.

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3. Look for a solution of the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}$ then $\lambda\mathbf{w} = \lambda\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = A\mathbf{w}$ and $0 = Det\begin{bmatrix} \frac{1}{2} - \lambda & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} - \lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$. The corresponding eigenvectors are $\lambda_1 = 2$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 = -1, \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$. There is one positive and one negative eigenvalue so the origin is a saddle point and trajectories are hyperbolas going from the asymptot $y = -x$ toward the asymptot $y = x$. The origin $(x = 0, y = 0)$ is an unstable equilibrium point since $\lambda_1 = 2 > 0$.

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4. The differences $u(x) = y_1(x) - y_2(x)$ and $w(x) = y_2(x) - y_3(x)$ are two linearly independent solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. The general solution is $y_{gen}(x) = y_{\hom}(x, A, B) + y_{part}(x) = Au(x) + Bw(x) + y_1(x)$. Functions $u(x), w(x)$ satisfy the homogeneous equations $u'' + p(x)u' + q(x)u = 0,\ w'' + p(x)w' + q(x)w = 0$. This is a set of two algebraic equations for 2 unknown functions $p(x), q(x)$. By linear independence of $u(x), w(x)$ there exist a solution $p(x) = -(uw'' - wu'')/(uw' - wu'),\ q(x) = (u'w'' - w'u'')/(uw' - wu')$. Knowing $p(x), q(x)$ we find $h(x) = y_1'' + p(x)y_1' + q(x)y_1$.

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5. Equilibrium points satisfy $xy - 2x = x(y-2) = 0, -xy + 6 - 2y = 0$. There are 2quilibrium points $(x_1 = 0, y_1 = 3)$ and $(x_2 = 1, y_2 = 2)$. The Jacobian is $J(x,y) = \begin{bmatrix} y-2 & x \\ -y & -x-2 \end{bmatrix}$. At $(x_1 = 0, y_1 = 3)$ $J(0,3) = \begin{bmatrix} 1 & 0 \\ -3 & -2 \end{bmatrix}$ has eigenvalues $\lambda_+ = 1 > 0, \lambda_- = -2 < 0$ and the point is unstable. At $(x_2 = 1, y_2 = 2)$

$J(1,2) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $0 = Det[J(1,2) - \lambda] = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$ gives the eigenvalues $\lambda_+ = -1 < 0, \lambda_- = -2 < 0$ and this point is stable. The condition $f(x,y) = x(y-2) = 0$ gives the nullclines $x = 0, y = 2$ with the vectofield parallel to y-axis. The condition $g(x, y) = -xy + 6 - 2y = 0$ defines a hiperbola $-xy + 6 - 2y = 0, x = \frac{6}{y} - 2$ where the vectorfield is parralel to x-axis. For every solution $x(t), y(t)$ of the dynamical system we have $\frac{d}{dt}(x + y - 3) = -2(x + y - 3)$ and therefore $(x(t) + y(t) - 3) = (x(0) + y(0) - 3)e^{-2t}$. So if the initiall point is on the line $(x(0) + y(0) - 3) = 0$ then $x(t), y(t)$ stays on the line $(x(t) + y(t) - 3) = 0$. Both equilibrium points belong to the line and the intervals between $(x_1 = 0, y_1 = 3)$ and $(x_2 = 1, y_2 = 2)$ and outside these points describe trajectories – 5 alltogether.

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6. From $x' = y = 0,\ y' = -x^3 + 4x = 0$ we get 3 equilibrium points (0,0), (2,0), (-2,0). By differentiating the first equation we see that $x'' = -x^3 + 4x = -\frac{\partial}{\partial x}(\frac{1}{4}x^4 - 2x^2) = -\frac{\partial}{\partial x}W(x)$ is a conservative potential Newton equation havings the energy integral $E = \frac{1}{2}x'^2 + W(x) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - 2x^2$ and $\dot{E}(x, y) = 0; E(2,0) = -4$. The function $V(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - 2x^2 + 4$ satisfies $V(x = 2, y = 0) = 0$. Define variables $x = 2 + z, y = w$. When $(x = 2, y = 0)$ then $(z = 0, w = 0)$ and the equilibrium point is in the origin now. $V(x = 2 + z, y = w) = \frac{1}{2}w^2 + \frac{1}{4}z^4 + 2z^3 + 4z^2$ is positive definite because for small z the term $4z^2 > 2z^3$. The directional derivative $\dot{V}_f(x, y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = (x^3 - 4x)y + y(-x^3 + 4x) = 0$.

So $\tilde{V}(z, w) = V(x = 2 + z, y = w) = \frac{1}{2}w^2 + \frac{1}{4}z^4 + 2z^3 + 4z^2 > 0$ when $(z, w) \neq (0,0)$ but close to $(0,0)$. $\tilde{V}(0,0) = 0$ and $\dot{\tilde{V}}_f(0,0) = 0$. $\tilde{V}(z, w)$ is a required Liapunov function for proving stability of (2,0) but not asymptotic stability. The algebraic equation $\tilde{V}(z, w) = \frac{1}{2}w^2 + \frac{1}{4}z^4 + 2z^3 + 4z^2 = const > 0$ defines closed trajectories around the equilibrium point (2,0).
