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Tentamen i Ordinära Differentialekvationer och Dynamiska System   2014-01-15, 8-13,
No aids allowed.   Kurskod: TATA71 (NMAC 26), Provkod: TEN 1

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find all solutions of the first order differential equation: $y' + 2xy = 2xy^2$
By solving this equation as a Bernoulli equation and also by the method of separation of variables. Show that both methods give the same set of solutions. Give a solution satisfying the initial condition $y(0) = 1$.

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2. Give a general solution of the linear system of equations: $\dot{x} = x + 3y,\ \dot{y} = 4x - 3y$
by finding eigenvalues and the eigenvectors first. Determine the fundamental matrix of this system either by using this solution or by calculating a suitable exponential of a matrix. Give a solution satisfying the initial conditions (IC) $x(0) = 2, y(0) = 4$. Check that this solution satisfies the system and the IC.

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3. Check that $y_1(x) = e^x$ solves the linear 2nd order differential equation: $(x+1)y'' - (x+2)y' + y = 0,\ x > 0$.
Find a second linearly independent solution $y_2(x)$. Derive a 1st order differential equation satisfied by a Wronskian $W[y_1, y_2]$ of two solutions $y_1(x), y_2(x)$ to the homogeneous linear 2nd order differential equation $y'' + p(x)y' + q(x)y = 0$. Control that the Wronskian of $y_1(x) = e^x$ and $y_2(x)$ satisfies this 1st order equation.

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4. Find equilibrium points and decide their stability for the dynamical system $\begin{cases} \dot{x} = x + 3y - 4, \\ \dot{y} = 4x - 3y - 1 \end{cases}$.
Find a general solution of this system and a solution satisfying IC $x(0) = 2, y(0) = -1$. Draw an approximate phase diagram of this system. Justify your picture.

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5. Consider the dynamical system: $\dot{x} = 3y - 5x - 4xy^2,\quad \dot{y} = -2x + 2x^2y$
Show, by finding a suitable Liapunov function of the form $V(x,y) = Ax^2 + By^2$ and by invoking the relevant Liapunov theorems, that the point $(x = 0, y = 0)$ is a stable equilibrium point and show that it is also asymptotically stable. What can you conclude about stability of $(0,0)$ by using the linear criterion of stability?

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6. Find an exact solution for the IVP: $y'(x) + 2xy(x) = x,\ y(0) = 1,$
Give an equivalent integral equation. Use the successive approximation formula to compute first 3 approximations $y_k(x),\ k = 1,2,3$ assuming that $y_0(x) = y(0)$. Compare $y_3(x)$ with the MacLaurin expansion of an exact solution. Which terms are different?

Lösningar tentamen TATA71 2014-01-15

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1. Assume that $y(x) \neq 0$ and divide the equation by $y^2(x)$. Then $y^{-2}y' + 2xy^{-1} = 2x$ and the substitution $z = y^{-1}$ turns this equation into a linear first order equation $z' - 2xz = -2x$ having an integrating factor $\exp(-x^2)$. So $[z\exp(-x^2)]' = -2x\exp(-x^2)$ and by integrating $z\exp(-x^2) = \exp(-x^2) + C$.
The solution $y^{-1} = z = 1 + C\exp(x^2)$ gives $y = 1/[1 + C\exp(x^2)]$ with $C \in \mathbf{R}$. The constant function $y(x) = 0$ is also a solution of the initial equation.
To separate variables rewrite the equation as $y' = 2x(y^2 - y) = 2x(y-1)y$. When $y \neq 0$ and $y - 1 \neq 0$ we get $2x = y'/(y-1)y = [\frac{1}{y-1} - \frac{1}{y}]y'$. By integrating both sides w.r.t. x we get $\ln\left|\frac{y-1}{y}\right| = x^2 + E$, then $\frac{y-1}{y} = \pm e^E \exp(x^2) = D\exp(x^2)$ with $D \neq 0$. By resolving for y we get $y = 1/[1 - D\exp(x^2)]$ where $D = \pm e^E \neq 0$ is an arbitrary real nonzero constant. $y(x) = 1 = const$ and $y(x) = 0$ also satisfy the initial equation. The solution $y(x) = 1 = const$ corresponds to choice of $D = 0$. Thus we get the same set of solutions when we take $D = -C \in \mathbf{R}$. The IC $1 = y(0) = 1/[1 + C\exp(x^2)]\big|_{x=0} = 1/(1+C)$ determines $C = 0$ and $y(x) = 1 = const$ is the required solution

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2. The matrix form is $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 3\\ 4 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$. Seek solutions of the form $\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\end{bmatrix}e^{\lambda t}$. The characteristic equation $0 = Det\begin{bmatrix}1-\lambda & 3\\ 4 & -3-\lambda\end{bmatrix} = \lambda^2 + 2\lambda - 15 = (\lambda - 3)(\lambda + 5)$ has two roots. The root $\lambda_1 = 3$ has the eigenvector $\begin{bmatrix}3\\ 2\end{bmatrix}$ and $\lambda_1 = -5$ has the eigenvector $\begin{bmatrix}1\\ -2\end{bmatrix}$. The general solution is $\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = A\begin{bmatrix}3\\ 2\end{bmatrix}e^{3t} + B\begin{bmatrix}1\\ -2\end{bmatrix}e^{-5t}$.
A solution satisfying the IC $\begin{bmatrix}x(0)=1\\ y(0)=0\end{bmatrix} = A\begin{bmatrix}3\\ 2\end{bmatrix}e^{3t} + B\begin{bmatrix}1\\ -2\end{bmatrix}e^{-5t}\Big|_{t=0} = \begin{bmatrix}3A+B\\ 2A-2B\end{bmatrix}$ is $\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix} = \begin{bmatrix}\frac{3}{4}e^{3t} + \frac{1}{4}e^{-5t}\\ \frac{1}{2}e^{3t} - \frac{1}{2}e^{-5t}\end{bmatrix}$
and satisfying the IC $\begin{bmatrix}x(0)=0\\ y(0)=1\end{bmatrix} = A\begin{bmatrix}3\\ 2\end{bmatrix}e^{3t} + B\begin{bmatrix}1\\ -2\end{bmatrix}e^{-5t}\Big|_{t=0} = \begin{bmatrix}3A+B\\ 2A-2B\end{bmatrix}$ is $\begin{bmatrix}x_2(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}\frac{3}{8}e^{3t} - \frac{3}{8}e^{-5t}\\ \frac{2}{8}e^{3t} + \frac{6}{8}e^{-5t}\end{bmatrix}$.
By the definition the fundamental matrix is $Y(t,0) = \begin{bmatrix}x_1(t) & x_2(t)\\ y_1(t) & y_2(t)\end{bmatrix} = \begin{bmatrix}\frac{3}{4}e^{3t} + \frac{1}{4}e^{-5t} & \frac{3}{8}e^{3t} - \frac{3}{8}e^{-5t}\\ \frac{1}{2}e^{3t} - \frac{1}{2}e^{-5t} & \frac{2}{8}e^{3t} + \frac{6}{8}e^{-5t}\end{bmatrix}$
and the solution satisfying IC $x(0) = 2, y(0) = 4$ is given by

$$Y(t,0)\begin{bmatrix}2\\ 4\end{bmatrix} = \begin{bmatrix}x_1(t) & x_2(t)\\ y_1(t) & y_2(t)\end{bmatrix}\begin{bmatrix}2\\ 4\end{bmatrix} = \begin{bmatrix}\frac{3}{4}e^{3t} + \frac{1}{4}e^{-5t} & \frac{3}{8}e^{3t} - \frac{3}{8}e^{-5t}\\ \frac{1}{2}e^{3t} - \frac{1}{2}e^{-5t} & \frac{2}{8}e^{3t} + \frac{6}{8}e^{-5t}\end{bmatrix}\begin{bmatrix}2\\ 4\end{bmatrix} = \begin{bmatrix}3e^{3t} - e^{-5t}\\ 2e^{3t} + 2e^{-5t}\end{bmatrix}$$

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3. The ansatz $y(x) = e^x u(x)$ gives $0 = (x+1)y'' - (x+2)y' + y = e^x[(x+1)u'' + xu']$. So $w(x) = u'(x)$ satisfies a linear 1st order equation $(x+1)w' + xw = 0$. In the canonical form we have $w' + \frac{x}{x+1}w = 0$. So $\int \frac{x}{x+1}dx = \int (1 - \frac{1}{x+1})dx = x - \ln|x+1|$. The integrating factor is $\exp(x - \ln|x+1|) = \frac{1}{x+1}e^x$ and $(\frac{1}{x+1}e^x w)' = 0$.
Take $w = (x+1)e^{-x}$. Then $u = \int w'(x)dx = -(x+2)e^{-x} + C$ and $y_2(x) = e^x u(x) = -(x+2) + Ce^x$.

A simplest 2nd solution is $y_2(x) = (x+2)$ when $C = 0$. Substitution into the equation confirms the result. The Wronskian satisfies the equation $W' = -p(x)W$ for the 2nd order equation written in the standard form $y'' + p(x)y' + q(x)y = 0$. Here $p(x) = -\frac{x+2}{x+1}$. $W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x & x+2 \\ e^x & 1 \end{vmatrix} = -(x+1)e^x$ and

$W' = [-(x+1)e^x]' = -(x+2)e^x = -(-\frac{x+2}{x+1})[-(x+1)e^x] = -(-\frac{x+2}{x+1})W$.

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4. Equations $0 = \dot{x} = x + 3y - 4,\ 0 = \dot{y} = 4x - 3y - 1$ give $(x_0 = 1, y_0 = 1)$ as an equilibrium point . The Jacobian matrix $J = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$ is constant. From $0 = Det(J - \lambda) = \begin{bmatrix} 1-\lambda & 3 \\ 4 & -3-\lambda \end{bmatrix} = \lambda^2 + 2\lambda - 15 = (\lambda + 5)(\lambda - 3)$ eigenvalues are $\lambda_+ = 3, \lambda_- = -5$ and the equilibrium (1,1) is unstable since $\lambda_+ = 3 > 0$. A general solution is a sum of the general solution of the homogeneous system and of a particular solution of the inhomogeneous system. A constant ansatz for a particular solution gives a constant solution $(x_p = 1, y_p = 1)$ that describes the equilibrium point.

Eigenvectors corresponding to the eigenvalues $\lambda_+ = 3, \lambda_- = -5$ are $w_+ = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $w_- = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{3t} + B\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-5t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From the initial condition $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} + B\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A = 0, B = 1$

The solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-5t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. New variables $u(t) = x(t) - 1, w(t) = y(t) - 1$ satisfy the homogeneous linear system $\dot{u} = u + 3w, \dot{w} = 4u - 3w$ of problem 2. So $\begin{bmatrix} u(t) = x(t) - 1 \\ w(t) = y(t) - 1 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{3t} + B\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-5t}$ is the general solution. To draw a phase diagram we see that straight lines $r\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{3t}, s\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-5t}$ consist of three trajectories each: the equilibrium point $(0,0)$ and two trajectories on the either side of $(0,0)$.

On the line $r\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{3t}$ we have two outgoing trajectories and on the line $r\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-5t}$ we have two trajectories approaching the equilibrium $(u = 0, w = 0)$. All remaining trajectories have hyperbolic shape. They are directed consistently with the half-line trajectories along $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and they approach these lines when $t \to \pm\infty$. The phase diagram in variables (x, y) has the same form with an equilibrium point shifted to (1,1).

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5. $(0,0)$ is an equilibrium point for the system $\dot{x} = 3y - 5x - 4xy^2$, $\dot{y} = -2x + 2x^2y$.

$\dot{V}(x,y) = 2Ax\dot{x} + 2By\dot{y} = 2Ax(3y - 5x - 4xy^2) + 2By(-2x + 2x^2y) = (6A - 4B)xy + (4B - 8A)x^2y^2 - 10Ax^2$.

Choose $A = 2, B = 3$. Then the function $V(x,y) = 2x^2 + 3y^2$ is a positive definite Liapunov function having $V(0,0) = 0$. It has negative semidefinite derivative $\dot{V}(x,y) = -4x^2y^2 - 20x^2 \le 0$ and $(0,0)$ is Liapunov stable.

For showing asymptotic stability study the set $M = \{(x,y) : 0 = \dot{V}(x,y) = -4x^2y^2 - 20x^2\} = \{(x = 0, y)\}$.

It consits of whole y-axis. If a solution satisfies $x(t) = 0$ then $\dot{y} = -2x + 2x^2y = 0$ and $0 = \dot{x} = 3y - 4x(t)y^2 = 3y$, implies that also $y(t) = 0$. So only the constant solution $(x(t) = 0, y(t) = 0)$ belongs to M and by the Liapunov theorem the point $(0,0)$ is also asymptotically stable.

The Jacobian $J_0 = J(0,0) = \begin{bmatrix} -5 & 3 \\ -2 & 0 \end{bmatrix}$ give $Det\begin{bmatrix} -5-\lambda & 3 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3)$ so eigenvalues are $\lambda_1 = -3, \lambda_2 = -2$. By the linear criterion of stability the point $(0,0)$ is asymptotically stable.

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6. Equation $y'(x) + 2xy(x) = x$ is both linear and separable if rewritten as $y'(x) = x(1-2y(x))$.

The linear equation has an integrating factor $\exp(x^2)$. So $[\exp(x^2)y(x)]' = x\exp(x^2)$ and $\exp(x^2)y(x) = \int x\exp(x^2)dx = \frac{1}{2}\exp(x^2) + C$. The general solution is $y(x) = \frac{1}{2} + C\exp(-x^2)$ and $y(x) = \frac{1}{2} + \frac{1}{2}\exp(-x^2)$ satisfies the initial condition $y(0) = 1$.

By integrating $y'(x) = x(1-2y(x)) = f(x, y(x))$ we get the integral equation

$y(x) = y(0) + \int_0^x f(s, y(s))ds = 1 + \int_0^x s(1-2y(s))ds$. The successive approximation formula reads

$y_k(x) = 1 + \int_0^x s(1-2y_{k-1}(s))ds \quad k = 1,2,3,4$. The first approximations are

$$y_1(x) = 1 + \int_0^x s(1-2y_0(s))ds = 1 + \int_0^x (-s)ds = 1 - \tfrac{1}{2}x^2$$

$$y_2(x) = 1 + \int_0^x s(1-2y_1(s))ds = 1 + \int_0^x s(-1+s^2)ds = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4$$

$$y_3(x) = 1 + \int_0^x s(1-2y_2(s))ds = 1 + \int_0^x s(-1+s^2-\tfrac{1}{2}s^4)ds = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{12}x^6$$

The MacLaurin expansion of the exact solution is

$$y(x) = \tfrac{1}{2} + \tfrac{1}{2}\exp(-x^2) = [z = -x^2] = \tfrac{1}{2}(1 + 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + O(z^4)) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{12}x^6 + O(x^8).$$

It has the same first 4 terms as $y_3(x)$.