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Tentamen i Ordinära Differentialekvationer och Dynamiska System 2014-04-23, kl. 14-19
No aids allowed. Kurskod: TATA71 (NMAC 26), Provkod: TEN 1

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find all equilibrium points and decide their stability for the logistic type equation $\frac{d}{dt}N = r[1-\frac{N}{K}]N - rN^2$ where r, K are positive constants. Find a general solution of this equation, a solution of the IC $N(0) = N_0$ and find a limit of this solution when $t \to \infty$.

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2. Give a general solution in a vector form for the linear system of equations $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and use it for drawing an approximate phase space diagram of this system. Determine a fundamental matrix of this system and a solution with IC $x(0) = 2, y(0) = -1$.

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3. Find a second linearly independent solution $y_2(x)$ of the differential equation: $x^2y'' - 5xy' + 9y = 0,\ x > 0$ when the solution $y_1(x) = x^3$ is known. Give a general solution of this equation. Confirm your result by choosing as independent variable $t = \ln x$ and by transforming this equation into a linear 2nd order differential equation with constant coefficients.

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4. Find all equilibrium points of the dynamical system $\dot{x} = y,\ \dot{y} = -x - y - z^2,\ \dot{z} = -z - xz$.
Decide stability of the equilibrium point in the origin $(0,0,0)$.

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5. Show that (0,0) is an asymptotically stable equilibrium point for the dynamical system

$$x' = -3x - 4y + 4y^2, \quad y' = -5y + x - xy$$

by taking the ansatz $V = ax^{2m} + by^{2n},\ m, n \in \mathbf{N}$ for a Liapunov function. Confirm this result by using the linear criterion of stability.

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6. Find a continuously differentiable function $y(x)$ that solves the integral equation

$$y(x) = 1 + \int_x^0 \frac{1}{s^2+1} y^2(s)ds$$

Lösningar tentamen TATA71 2014-04-23, 14-19,

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1. The equation can be rewritten as a logistic equation with $\widetilde{K}(E) = \frac{K}{1+K}$ since $\dot{N} = r[1-\frac{N}{K}]N - rN^2 = rN - (\frac{r}{K}+r)N^2 = r[1-N/(\frac{K}{1+K})]N = r[1-N/\widetilde{K}]N = f(N)$. Equilibrium points are $N_0 = 0$ and $N_1 = \widetilde{K} = \frac{K}{1+K}$. Since $f'(N) = (rN - \frac{r}{\widetilde{K}}N^2)' = r - 2\frac{r}{\widetilde{K}}N$, we have $f'(N_0) = r > 0$ and $N_0 = 0$ is unstable. $f'(N_1) = -r < 0$ is negative and $N_1 = \widetilde{K} = \frac{K}{1+K}$ is stable. By separating variables in the the equation $\dot{N} = r[1-N/\widetilde{K}]N$ we find the solution $N(t) = \widetilde{K}/[1+C\exp(-rt)]$ where C is an integration constant. From the IC $N(0) = N_0$ we get $N(t) = \widetilde{K}/[1+(\frac{\widetilde{K}}{N_0}-1)e^{-rt}] \xrightarrow{t\to\infty} \widetilde{K}$.

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2. $\det\begin{bmatrix} 5-\lambda & 8 \\ -3 & -5-\lambda \end{bmatrix} = 0$ gives eigenvalues $\lambda_1 = 1, \lambda_2 = -1$. The corresponding eigenvectors are $w_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 2 \\ -1 \end{bmatrix}e^t + B\begin{bmatrix} 4 \\ -3 \end{bmatrix}e^{-t}$.

The origin is an unstable equilibrium of the saddle point type. The line through the origin along the eigenvector w_1 contains two outgoing trajectories and the line along the eigenvector w_2 contains two ingoing trajectories. All remaining trajectories have a hyperbolic shape. They are directed consistently with the half-line trajectories along w_1, w_2 and they approach these lines when $t \to \pm\infty$.

A solution that satisfies the IC $\begin{bmatrix} x(0)=1 \\ y(0)=0 \end{bmatrix} = A\begin{bmatrix} 2 \\ -1 \end{bmatrix}e^t + B\begin{bmatrix} 4 \\ -3 \end{bmatrix}e^{-t}\Bigg|_{t=0} = \begin{bmatrix} 2A+4B \\ -A-3B \end{bmatrix}$ is $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{-t} \\ -\frac{3}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}$

and a solution satisfying the IC $\begin{bmatrix} x(0)=0 \\ y(0)=1 \end{bmatrix} = A\begin{bmatrix} 2 \\ -1 \end{bmatrix}e^t + B\begin{bmatrix} 4 \\ -3 \end{bmatrix}e^{-t}\Bigg|_{t=0} = \begin{bmatrix} 2A+4B \\ -A-3B \end{bmatrix}$ is $\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 4e^t - 4e^{-t} \\ -2e^t + 3e^{-t} \end{bmatrix}$.

By the definition the fundamental matrix is $Y(t,0) = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{-t} & 4e^t - 4e^{-t} \\ -\frac{3}{2}e^t + \frac{3}{2}e^{-t} & -2e^t + 3e^{-t} \end{bmatrix}$.

A solution satisfying IC $x(0) = 2, y(0) = -1$ is given by

$$Y(t,0)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{-t} & 4e^t - 4e^{-t} \\ -\frac{3}{2}e^t + \frac{3}{2}e^{-t} & -2e^t + 3e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}e^t$$

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3. The ansatz $y(x) = x^3u(x)$ gives $0 = x^2y'' - 5xy' + 9y = x^4(xu'' + u') = [w = u'] = x^4(xw' + w)$. $xw' + w = 0$ has the solution $w = C/x$ and $u = C\ln|x| + D$. We choose $y_2(x) = x^3\ln x$. A general solution is $y(x,A,B) = Ay_1(x) + By_2(x) = Ax^3 + Bx^3\ln x$. To change variables differentiate $y(x) = \widetilde{y}(t = \ln x)$ according to the chain rule. $\frac{dy(x)}{dx} = \frac{d\widetilde{y}(t=\ln x)}{dt}\frac{dt}{dx} = \frac{d\widetilde{y}(t)}{dt}\frac{1}{x}$, $\frac{d^2y(x)}{dx^2} = (\frac{d}{dt}\frac{d\widetilde{y}(t)}{dt})\frac{dt}{dx}\frac{1}{x} + \frac{d\widetilde{y}(t)}{dt}\frac{d}{dx}(\frac{1}{x}) = \frac{d^2\widetilde{y}(t)}{dt^2}(\frac{1}{x})^2 - \frac{d\widetilde{y}(t)}{dt}(\frac{1}{x})^2$. After substituting into the initial equation we get $0 = x^2y'' - 5xy' + 9y = \widetilde{y}'' - \widetilde{y}' - 5\widetilde{y}' + 9\widetilde{y} = \widetilde{y}'' - 6\widetilde{y}' + 9\widetilde{y}$. The characteristic equation $0 = \lambda^2 - 6\lambda + 9$ has a double root $\lambda_\pm = 3$. The general solution is

$\widetilde{y}(t) = Ae^{3t} + Be^{3t}t = Ae^{3\ln x} + Be^{3\ln x}\ln x = Ax^3 + Bx^3\ln x$.

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4. Equations $0 = \dot{x} = y$, $0 = \dot{y} = -x - y - z^2$, $0 = \dot{z} = -z - xz$ give 3 points $(x_0 = 0, y_0 = 0, z_0 = 0)$ and

$(x_1=-1, y_1=0, z_1=1)$, $(x_2=-1, y_2=0, z_2=-1)$. The Jacobian is $J=\begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -2z \\ -z & 0 & -1-x \end{bmatrix}$

$Det[J(0,0,0)-\lambda Id] = Det\begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} = -(\lambda+1)(\lambda^2+\lambda+1)=0$ has three solutions, $\lambda_0=-1$

and $\lambda_{\pm}=\frac{1}{2}[-1\pm i\sqrt{3}]$. All eigenvalues have real part negative so the origin $(0,0,0)$ is asymptotically stable.

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5. Equations $x'=-3x-4y+4y^2=0$, $y'=-5y+x-xy=0$ have only one real valued solution (0,0).
$V=\frac{1}{4}x^2+y^2$ is a suitable positive definite Liapunov function. The directional derivative is
$\dot{V}=\frac{d}{dt}(\frac{1}{4}x^2+y^2)=\frac{1}{2}x\dot{x}+2y\dot{y}=\frac{1}{2}x(-3x-4y+4y^2)+2y(-5y+x-xy)=-\frac{3}{2}x^2-10y^2<0$. So by the Liapunov theorem (0,0) is an asymptotically stable point. The Jacobian $J(0,0)=\begin{bmatrix} -3 & -4 \\ 1 & -5 \end{bmatrix}$ has eigenvalues $\lambda_{\pm}=-4\pm 2\sqrt{3}i$. $\mathrm{Re}\lambda_{\pm}=-4<0$ and by the linear criterion of stability we can conclude that (0,0) is asymptoticaly stable.

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6. By taking derivative of the integral equation we get a first order linear differential equation
$y'=(-\frac{1}{1+x^2})y^2$ and by setting $x=0$ in the integral equation we obtain the IC $y(0)=1$.
This is a Bernoulli type equation. The substitution $z(x)=1/y(x)$, $y(x)\neq 0$ gives $y'(x)=-z'(x)/z(x)^2$ and $z'=\frac{1}{1+x^2}$. So $z(x)=\arctan x+C$ and $y(x)=1/(\arctan x+C)$. From the IC $1=y(0)=\frac{1}{C}$ the solution is $y(x)=1/(\arctan x+1)$. Substitution of this solution into the integral equation gives

$$1+\int_x^0 \frac{1}{s^2+1}y^2(s)ds = 1+\int_x^0 \frac{1}{s^2+1}\frac{1}{(\arctan s+1)^2}ds = 1+\left[\frac{-1}{(\arctan s+1)}\right]_x^0 = 1-1+\frac{1}{(\arctan x+1)} = y(x)$$