Tekniska Högskolan i Linköping, Matematiska Institutionen
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Tentamen i Ordinära Differentialekvationer och Dynamiska System 2015-01-14 kl. 8 - 13
No aids allowed. Kurskod: TATA71, Provkod: TEN

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find a general solution for the first order differential equation: $(x^2+1)y' + xy = x(6-2y)$.
a) Through the method of separation of variable b) By solving through the integrating factor method.
Show that it is the same general solution meaning that domains of the integration constant are the same.
c) Give a solution for the following initial conditions: (i) $y(0)=1$, (ii) $y(1)=2$.

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2. Find a general solution of the linear system of equations $\dot{x} = 3x - y$, $\dot{y} = 6x - 4y$ in two ways.
a) By reducing this system to a second order equation for $x(t)$ and solving it for $x(t), y(t)$.
b) By solving through the vector ansatz for a solution $(x(t), y(t))^t = (w_1, w_2)^t e^{\lambda t}$.
Show that both methods give the same general solution. Find a fundametal matrix of solutions for this system.

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3. For a linear inhomogeneous equation of second order $y'' + ay' = 3a$ with $0 \neq a \in R$ the constant function $y_1(x) = 1$ is a solution of the homogenous equation.
a) Find the second solution $y_2(x)$ of the homogeneous equation through the method of variation of constant.
b) Find a particular solution $y_p(x)$ of the inhomogeneous equation through the Lagrange ansatz $y_p(x) = u(x)y_1 + v(x)y_2$. c) Find a general solution of this inhomogeneous equation with constant coefficients in a usual way to confirm that you have got corrects answers in a) and b).

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4. For a linear, inhomogeneous 2nd order differential equation $y'' + p(x)y' + q(x)y = h(x)$ the function $u(x)$ is a solution of the homogeneous equation and $y_r(x)$, $y_s(x)$ are two known particular solutions of the inhomogeneous equation. All three functions $u(x)$, $y_r(x)$, $y_s(x)$ are linearly independent.
a) Express a general solution of this equation through the solutions $u(x), y_r(x)$ and $y_s(x)$.
b) Reconstruct coefficients of the equation $p(x), q(x), h(x)$ from these particular solutions and their derivatives.

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5. A dynamical system is given by equations $\dot{x} = \frac{2xy}{y+1} - x$, $\dot{y} = -\frac{xy}{y+1} + 2 - y$
Show that the system has 2 equilibrium points and decide their linear stability. Draw all nullclines and direction of the vectorfield on the nullclines. Show that if $x(t), y(t)$ solve the dynamical system then $\frac{d}{dt}(x+2y-4) = -(x+2y-4)$. Solve the last equation to show that if the initial condition is on the line $x(0)+2y(0)-4=0$ then the solution stays on this line $(x(t)+2y(t)-4)=0$. Draw the line and describe all 5 trajectories belonging to this line .

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6. Formulate the Liapunov theorem on asymptotic stability of an equilibrium point of an autonomous dynamical system. Show by taking the ansatz $V = ax^{2m} + by^{2n}$, $m, n \in \mathbf{N}$ for a Liapunov function that (0,0) is an asymptotically stable equilibrium point for the system $\dot{x} = -x + 4y^4$, $\dot{y} = -y - xy$ and therefore all solutions go to (0,0) when $t \to \infty$. Confirm this by using the linear criterion of stability.

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Solutions ODE´s (TATA71) 2015-01-14

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1. To separate variables rewrite the equation as $(x^2+1)y' = -3x(y-2)$ and when $y-2 \neq 0$ we get $\frac{1}{y-2}y' = -\frac{3x}{x^2+1}$ then $\ln|y-2| = -\frac{3}{2}\ln|x^2+1| + C$ and $\ln\left|(y-2)(x^2+1)^{\frac{3}{2}}\right| = C$. By exponentiating and taking away the absolute value $(y-2)(x^2+1)^{\frac{3}{2}} = \pm e^C = D \neq 0$. So the solution is $y = 2 + D(x^2+1)^{-\frac{3}{2}}$ where $D \neq 0$. But the condition $y-2 \neq 0$ excludes the constant solution $y = 2$. So in the general solution the constant $D \in \mathbf{R}$ can take an arbitrary real value. As a linear equation it has the standard form $y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$. An integrating factor is $\exp(\int (\frac{3x}{x^2+1})dx) = \exp(\frac{3}{2}\ln|x^2+1|) = (x^2+1)^{3/2}$ By multiplying with it both sides of the equation we obtain $((x^2+1)^{3/2}y)' = (x^2+1)^{3/2}y' + 3x(x^2+1)^{1/2}y = 6x(x^2+1)^{1/2} = 2((x^2+1)^{3/2})'$. By integrating both sides $(x^2+1)^{3/2}y = 2(x^2+1)^{3/2} + D$ and $y = 2 + D(x^2+1)^{-3/2}$ with $D \in \mathbf{R}$.

For (i) $1 = y(0) = 2 + D$ and $y(x) = 2 - (x^2+1)^{-3/2}$ (ii) $2 = y(1) = 2 + D(1^2+1)^{-3/2}$ and $y(x) = 2$.

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2. By deriving $\ddot{x} = 3\dot{x} - \dot{y} = 3\dot{x} - 6x + 4y = 3\dot{x} - 6x + 4(3x - \dot{x}) = 6x - \dot{x}$ and $\ddot{x} + \dot{x} - 6x = 0$

$0 = \ddot{x} + \dot{x} - 6x = [x = e^{\lambda x}] = e^{\lambda x}(\lambda^2 + \lambda - 6) = e^{rx}(\lambda+3)(\lambda-2)$. The general solution is $x(t) = Ae^{-3t} + Be^{2t}$ and $y = 3x - \dot{x} = 3(Ae^{-3t} + Be^{2t}) - (-3Ae^{-3t} + 2Be^{2t}) = 6Ae^{-3t} + Be^{2t}$.

The ansatz $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}$ gives $\lambda \mathbf{w} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = A\mathbf{w}$ and

$0 = Det \begin{bmatrix} 3-\lambda & -1 \\ 6 & -4-\lambda \end{bmatrix} = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$. The corresponding eigenvalues and eigenvectors are

$\lambda_1 = -3$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\lambda_2 = 2$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{-3t} + B\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$.

A solution satisfying IC $x(0) = 1, y(0) = 0$ is $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -e^{-3t} + 6e^{2t} \\ -6e^{-3t} + 6e^{2t} \end{bmatrix}$, and satisfying IC

$x(0) = 0, y(0) = 1$ is $\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{-3t} - e^{2t} \\ 6e^{-3t} - e^{2t} \end{bmatrix}$. The fundamental matrix is

$$X = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -e^{-3t} + 6e^{2t} & e^{-3t} - e^{2t} \\ -6e^{-3t} + 6e^{2t} & 6e^{-3t} - e^{2t} \end{bmatrix}$$

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3. Variation of constant means that we look for a second solution in form $y = u(x)y_1 = u(x)$. So $u(x)$ satisfies $u'' + au' = 0$ and $w = u'$ satisfying $w' + aw = 0$ is $w = e^{-ax}$. Thus $u' = e^{-ax}$, $u = -\frac{1}{a}e^{-ax}$ and we can take $y_2(x) = e^{-ax}$.

To calculate a particular solution we take the ansatz $y(x) = u(x)y_1 + v(x)y_2$ and then $y' = uy_1' + vy_2' + u'y_1 + v'y_2$. By the auxiliary condition we set $0 = u'y_1 + v'y_2 = u' + v'e^{-ax}$ and then $y'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'$. From the inhomogeneous equation $3a = y'' + ay' = uy_1'' + vy_2'' + u'y_1' + v'y_2' + a(uy_1' + vy_2') = u'y_1' + v'y_2' = v'y_2' = v'(-ae^{-ax})$. So $v' = -3e^{ax}, v = -\frac{3}{a}e^{ax}$. From the auxiliary condition $u' = -v'e^{-ax} = 3$, $u = 3x + C$ and $y_p(x) = u(x)y_1 + v(x)y_2 = 3x + (C - \frac{3}{a})$. Since $(C - \frac{3}{a})$ is proportional to a homogenious solution $y_1(x) = 1$

the simplest form of the particular solution is $y_p(x) = 3x$. The general solution is $y_{gen} = Ay_1 + By_2 + y_p =$ $= A + Be^{-ax} + 3x$. The standard method of solving this equation $0 = y'' + ay' = [y = e^{rx}] = e^{rx}(r^2 + ar) = e^{rx}(r+a)r$ gives $y_h = A + Be^{-ax}$. The ansatz $y_p = bx + c$ give the same particular solution $y_p = 3x$.

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4. The difference $w(x) = y_r(x) - y_s(x)$ is a second independent solution of the homogeneous equation. A general solution is $y_{gen}(x) = y_{\hom}(x, A, B) + y_{part}(x) = Au(x) + B[y_r(x) - y_s(x)] + y_s(x)$.Functions $u(x), w(x)$ satisfy the homogeneous equations $u'' + p(x)u' + q(x)u = 0$, $w'' + p(x)w' + q(x)w = 0$. We read this as a set of two algebraic equations for 2 unknown functions $p(x), q(x)$. By linear independence of $u(x), w(x)$ there exist a solution $p(x) = -(w'' - u'')/(w' - u')$, $q(x) = (u'w'' - w'u'')/(w' - u')$. Knowing $p(x), q(x)$ we find $h(x) = y_s'' + p(x)y_s' + q(x)y_s$.

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5. Equilibrium points satisfy $0 = \frac{2xy}{y+1} - x = (\frac{2y}{y+1} - 1)x = f(x,y)$, $0 = -\frac{xy}{y+1} + 2 - y = g(x,y)$. There are 2 equilibrium points $(x_1 = 0, y_1 = 2)$ and $(x_2 = 2, y_2 = 1)$. The Jacobian is $J(x,y) = \begin{bmatrix} \frac{2y}{y+1} - 1 & \frac{2x}{(y+1)^2} \\ -\frac{y}{y+1} & -\frac{x}{(y+1)^2} - 1 \end{bmatrix}$.

At $(x_1 = 0, y_1 = 2)$ $J(0,2) = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{3} & -1 \end{bmatrix}$ has eigenvalues $\lambda_+ = \frac{1}{3} > 0, \lambda_- = -1 < 0$ and the point is unstable.

At $(x_2 = 2, y_2 = 1)$ $J(2,1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$, $0 = Det[J(1,1) - \lambda] = \begin{bmatrix} -\lambda & 1 \\ -\frac{1}{2} & -\frac{3}{2} - \lambda \end{bmatrix} = \lambda^2 + \frac{3}{2}\lambda + \frac{1}{2} = (\lambda + 1)(\lambda + \frac{1}{2})$ gives the eigenvalues $\lambda_+ = -1 < 0, \lambda_- = -\frac{1}{2} < 0$ and this point is stable. The condition $f(x,y) = (\frac{2y}{y+1} - 1)x = 0$ gives the nullclines $x = 0, y = 1$ with the vectofield parallel to y-axis. The condition $g(x,y) = -\frac{xy}{y+1} + 2 - y = 0$ defines a hiperbola $x = (\frac{2}{y} - 1)(y+1) = \frac{2}{y} + 1 - y$ where the vectorfield is parallel to x-axis.

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6. Equations $\dot{x} = -x + 4y^4 = 0$, $\dot{y} = -y - xy = 0$ have only one real valued solution (0,0). The directional derivative is
$\dot{V} = \frac{d}{dt}(ax^2 + by^4) = 2ax\dot{x} + 4by^3\dot{y} = 2ax(-x + 4y^4) + 4by^3(-y - xy) = -2ax^2 - 4by^4 + 8axy^4 - 4bxy^4$.
It is negative definite for $a = 1, b = 2$ and then $\dot{V} = -2x^2 - 8y^4 < 0$. By the Liapunov theorem (0,0) is asymptotically stable.

The Jacobian $J(0,0) = \left. \begin{bmatrix} -1 & 16y^3 \\ -y & -1-x \end{bmatrix} \right|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, has a double real eigenvalue $\lambda_\pm = -1$ and from the linear criterion we can conclude that (0,0) is asymptoticaly stable.
