Tekniska Högskolan i Linköping, Matematiska Institutionen
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Tentamen i Ordinära Differentialekvationer och Dynamiska System 2015-04-08 kl. 14 - 19
No aids allowed. Kurskod: TATA71, Provkod: TEN

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find a general solution for $xy' = 2y$ by solving it as a linear equation and as a separable equation. Show that it is the same general solution meaning that domains of the integration constant are the same. Give a solution for the following initial conditions: (i) $y(1) = 2$, (ii) $y(0) = 1$.

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2. A body moves horizontally through a resisting medium, whose resistance is proportional to the fourth power of the velocity $v(t)$, according to the Newton law $\frac{dv}{dt} = -kv^4$. Find the velocity $v(t)$ if the IC is $v(t=0) = W$. Show that the distance traversed in time t is $x(t) = \frac{1}{2k}(3kt + W^{-3})^{2/3} - \frac{1}{2k}W^{-2}$ if the body starts at $x(0) = 0$ with velocity $v(0) = W$.

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3. Find a general solution in real form for the dynamical system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and draw an approximate phase diagram of trajectories. Decide stability of the equilibrium point $(0,0)$.

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4. Find through the Lagrange´ method a particular solution of the differential equation $y'' + 3y' + 2y = 2$. Confirm your solution by taking a polynomial ansatz for $y_p(x)$. Give a solution satisfying the initial value problem $y(0) = 0$, $y'(0) = 0$.

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5. Formulate the Liapunov theorem on asymptotic stability of an equilibrium point of an autonomous dynamical system. Show by taking the ansatz $V = ax^2 + by^4$, for a Liapunov function that (0,0) is an asymptotically stable equilibrium point for the system

$$x' = -x + 4y^4, \qquad y' = -y - xy$$

and therefore all solutions have the limit (0,0) when $t \to \infty$. Confirm this by applying the linear criterion of stability.

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6. Show that a continuously differentiable function $y(x)$ is a solution to the initial value problem $\frac{d}{dx} y(x) = f(x,y)$, $y(0) = b$ with a continuous function $f(x,y)$, (x, y, b, $f(x,y)$ are real) if and only if $y(x)$ is a continuous solution of the integral equation $y(x) = b + \int_0^x f(s, y(s))ds$. Use this results to solve the integral equations $y(x) + 2 = \int_0^x s(y(s)^2 + 2y(s))ds$. Check that the solution satisfies the integral equation.

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Solutions ODE´s (TATA71) 2015-04-08

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1. In the standard form the equation $y' + (-\frac{2}{x})y = 0$ is defined only for $x \neq 0$. The integrating factor $k(x) = \exp(-\int \frac{2}{x} dx) = x^{-2}$ gives $0 = x^{-2}y' - 2x^{-3}y = (x^{-2}y)'$. So $y(x) = Cx^2, C \in R$. When we separate variables $y'/y = 2/x$ we have to assume that $y(x) \neq 0$ and $x \neq 0$. By integrating both sides w.r.t. x we get $2\ln|x| + D = \int (2/x)dx = \int (y'/y)dx = [dy = y'dx] = \int (1/y)dy = \ln|y|$.

By taking exponent of both sides $|y(x)| = \exp(\ln|y(x)|) = \exp(2\ln|x| + D) = e^D x^2$ we get $y(x) = \pm e^D x^2 = Cx^2$ where $C = \pm e^D \neq 0$ is an arbitrary nonzero constant. But $y(x) = 0$ also satisfies $xy' = 2y$ and both solutions are given together by the same formula $y(x) = Cx^2, C \in R$.

The initial condition (i) $2 = y(1) = Cx^2\big|_{x=1} = C$ gives $y(x) = 2x^2$, but (ii) $1 = y(0) = Cx^2\big|_{x=0} = 0$ gives a contradiction since the standard equation is not defind for $x = 0$.

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2. By separating variables we get $v^{-4}\frac{dv}{dt} = -k$ if $v(t) \neq 0$. After integrating both sides w.r.t. time t we get $-\frac{1}{3}v^{-3} = \int v^{-4}\frac{dv}{dt} = \int (-k)dt = -kt - C$, $v(t) = 1/(3kt + 3C)^{1/3}$ and from the initial condition $C = \frac{1}{3}W^{-3}$. By integrating the velocity $v(t) = 1/(3kt + W^{-3})^{1/3}$,

$x(t) = \int v(t)dt = \int (3kt + 3C)^{-1/3}dt = \frac{1}{2k}(3kt + W^{-3})^{2/3} + D$ where $D = -\frac{1}{2k}W^{-2}$.

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3. $\det\begin{bmatrix} 1-\lambda & -5 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 25 = 0$ gives eigenvalues $\lambda_\pm = 3 \pm 4i$,. The complex eigenvector corresponding to $\lambda_+ = 3 + 4i$ is $w_1 = e^{3t}\begin{bmatrix} 1-2i \\ -2 \end{bmatrix}(\cos 4t + i\sin 4t)$. The general solution is

$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = Ae^{3t}\begin{bmatrix} \cos 4t + 2\sin 47 \\ -2\cos 4t \end{bmatrix} + Be^{3t}\begin{bmatrix} -2\cos 4t + \sin 4t \\ -2\sin 4t \end{bmatrix}$. It describes unwinding spirals going from $(0,0)$ to infinity. Since $\operatorname{Re}\lambda_\pm = \operatorname{Re}(3 \pm 4i) = 3 > 0$ the equilibrim point $(0,0)$ is unstable.

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4. The homogeneous equation $0 = y'' + 3y' + 2y = [y(x) = e^{\lambda x}] = e^{\lambda x}(\lambda^2 + 3\lambda + 2) = e^{\lambda x}(\lambda + 1)(\lambda + 2)$ gives two solutions $y_1(x) = e^{-x}, y_2(x) = e^{-2x}$ and the general solution of the homogeneous equation is $y_h(x) = Ae^{-x} + Be^{-2x}$. We seek $y_p(x) = u(x)y_1(x) + w(x)y_2(x) = u(x)e^{-x} + w(x)e^{-2x}$. From the inhomogeneous equation by using the auxiliary condition $u'(x)e^{-x} + w'(x)e^{-2x} = 0$ we get $u'(x)y_1'(x) + w'(x)y_2'(x) = u'(x)(-e^{-x}) + w'(x)(-2e^{-2x}) = 2$. Together with the auxiliary condition $u'(x)e^{-x} + w'(x)e^{-2x} = 0$, we get 2 algebraic equations for $u'(x), w'(x)$. This leads to the solution $y_p(x) = -y_1(x)\int_0^x y_2(s)\frac{h(s)}{W(s)}ds + y_2(x)\int_0^x y_1(s)\frac{h(s)}{W(s)}ds = 1 - 2e^{-x} + e^{-2x}$ where the $W(x)$ denotes the Wronskian function $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = -e^{-3x}$. This $y_p(x) = 1 - 2e^{-x} + e^{-2x}$ satisfies the initial conditions $y(0) = 0$, $y'(0) = 0$.

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5. Equations $x' = -x + 4y^4 = 0$, $y' = -y - xy = 0$ have only one real valued solution (0,0). The directional derivative is

$$\dot{V} = \frac{d}{dt}(ax^2 + by^4) = 2ax\dot{x} + 4by^3\dot{y} = 2ax(-x + 4y^4) + 4by^3(-y - xy) =$$

$-2ax^2 + 8axy^4 - 4by^4 - 4bxy^4 = [a = 1, b = 2] = -2x^2 - 8y^4 < 0$. It is negative definite for $(x, y) \neq 0$. $V = x^2 + 2y^4 > 0$ is a positive definite Liapunov function for $(x, y) \neq 0$. By the Liapunov theorem (0,0) is asymptotically stable.

The Jacobian $J(0,0) = \left.\begin{bmatrix} -1 & 16y^3 \\ -y & -1-x \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, has a double negative real eigenvalue $\lambda_1 = -1$ and from the linear criterion we can also conclude that (0,0) is asymptoticaly stable.

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6. By taking derivative of the integral equation we get a first order linear ODE $y' = x(y^2 + 2y)$ with the IC $y(0) = -2$. By separating variables (for $y(x) \neq 0$ and $y(x) \neq -2$) we get $y' / y(y+2) = x$. Observe that the constant functions $y(x) = 0$ and $y(x) = -2$ are also solutions. By integrating we get $\ln|y/(y+2)| = x^2 + C$ and $\frac{y}{y+2} = \pm e^C e^{x^2} = De^{x^2}$ with $D \neq 0$. So $y(x) = 2D\exp(x^2)/(1 - D\exp(x^2))$. When $D = 0$ we obtain solution $y(x) = 0$ and notice that $y(x) = 2D\exp(x^2)/(1 - D\exp(x^2)) \to -2$ when $D \to \infty$. The constant solution $y(x) = -2$ satisfies the given initial condition. By substituting this solution to the integral equation we get

$$0 = -2 + 2 = y(x) + 2 = \int_0^x s(y(s)^2 + 2y(s))ds = \int_0^x s((-2)^2 + 2(-2))ds = \int_0^x 0ds = [const]_0^x = 0.$$
