Tentamen i Ordinära Differentialekvationer och Dynamiska System 2016-01-13 kl. 8 - 13
No aids allowed. Kurskod: TATA71, Provkod: TEN

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 point and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

----

1. Find a general solution of the equation $y' = (1+y)/(x^2+x)$ by separating variables and by solving as a linear equation. Show that it is the same general solution meaning that domains of the integration constant are the same. Give a solution for the following initial conditions: (i) $y(-3) = -1$, (ii) $y(1) = -2$. For which values of x are solutions (i) and (ii) defined?

----

2. Find equilibrium points of the linear system of equations: $\dot{x} = x + 3y, \dot{y} = 4x - 3y$ and decide their stability. Confirm your result by calculating a general solution of this linear system of equations. Draw an approximate phase portrait for the system.

----

3. The 1-parameter family of functions $y(x) = Ae^x + x^2, A \in \mathbf{R}$ solves a linear 2nd order differential equation $(x+1)y'' - (x+2)y' + q(x)y = h(x)$. Determine the equation and find a general solution of this equation.

----

4. Find all equilibrium points for the dynamical system: $\dot{x} = y^2 - x^2$, $\dot{y} = x - 1$. Decide their linear stability and draw all nullclines and direction of the vectorfield on the nullclines.

----

5. Show that for all $a > 0$ the origin $(x = 0, y = 0)$ is an asymptotically stable equilibrium point for the dynamical system: $\dot{x} = y$, $\dot{y} = -4x - ay$
by finding a suitable Liapunov function depending quadratically on the dynamical variables. Confirm this result by using the linear criterion of stability.

----

6. Show that a continuously differentiable function $y(x)$ is a solution to the initial value problem $\frac{d}{dx} y(x) = f(x,y)$, $y(0) = b$ with a continuous function $f(x,y)$, ($x, y, b,$ f(x,y) are real) if and only if $y(x)$ is a continuous solution of the integral equation $y(x) = b + \int_0^x f(s, y(s))ds$. Use this result to solve the integral equation $y(x) + 2x - 1 = \int_0^x (y(s)^2 + 1)ds$. Check that the solution satisfies the integral equation.

Solutions Ordinära Differentialekvationer och Dynamiska System TATA71 2016-01-13

1. Equation $y' = (1+y)/(x^2+x) = (1+y)/(x+1)x$ is defined for $x \neq 0, x \neq -1$. For $y \neq -1$ by separating variables we get $\ln\left|\frac{x}{x+1}\right| + C = \int \frac{1}{x(x+1)}dx = \int \frac{y'}{y+1} y'\,dx = \ln|y+1|$, $C \in \mathbf{R}$. By solving this for y: $\ln\left|(y+1)\frac{x+1}{x}\right| = C$ then $(y+1)\frac{x+1}{x} = \pm e^C = D \neq 0$ and $y = \frac{Dx}{x+1} - 1$. But $y = -1$ is also a solution so the general solution is $y = \frac{Dx}{x+1} - 1$ with $D \in \mathbf{R}$. The equation rewitten as $y' - \frac{1}{x(x+1)} y = \frac{1}{x(x+1)}$ has an integrating factor $\frac{x+1}{x}$. Then $(\frac{x+1}{x} y)' = \frac{x+1}{x} y' - \frac{1}{x^2} y = \frac{1}{x^2}$ and $\frac{x+1}{x} y = -\frac{1}{x} + D$, so $y = \frac{(E+1)x}{x+1} - 1 = \frac{Dx}{x+1} - 1$ with $D, E \in \mathbf{R}$.
Solutions of initial conditions: (i) $y(x) = -1$, $-\infty < x < -1$, (ii) $y(x) = -\frac{3x+1}{x+1}$, $0 < x < \infty$.

----

2. The homogeneous system of linear equations $\dot{x} = x + 3y, \dot{y} = 4x - 3y$ has an equilibrium at (0,0).
For the vector equation $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ seek solutions of the form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$. The characteristic equation $0 = Det\begin{bmatrix} 1-\lambda & 3 \\ 4 & -3-\lambda \end{bmatrix} = (\lambda - 3)(\lambda + 5)$ has two roots. The equilibrium (0,0) is unstable since $\lambda_1 = 3 > 0$. The root $\lambda_1 = 3$ has the eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\lambda_1 = -5$ has the eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t} + B\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t}$. The straight lines $r\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t}, s\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t}$ consist of three trajectories each: the equilibrium point $(0,0)$ and two trajectories on the either side. On the line $r\begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t}$ we have two outgoing trajectories and on the line $s\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t}$ we have two ingoing trajectories approaching the equilibrium $(0,0)$. All remaining trajectories are hyperbola type curves directed consistently with the half-line trajectories along $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

----

3. Since the parameter A is arbitrary the function $y_p(x) = x^2$ is a particular solution of the inhomogeneous equation and $y_1(x) = e^x$ is a solution of the homogeneous equation. By setting into equation $0 = (x+1)y_1'' - (x+2)y_1' + q(x)y_1 = [y_1 = e^x] = e^x[-1 + q(x)]$ so $q(x) = 1$. By substituting $y_p(x)$ into the inhomogeneous equation

$h(x) = (x+1)y_p'' - (x+2)y_p' + y_p = (x+1)2 - (x+2)2x + x^2 = -x^2 - 2x + 2$ and a complete equation reads $(x+1)y'' - (x+2)y' + y = -x^2 - 2x + 2$. For finding a second solution take the ansatz $y(x) = e^x u(x)$. Then $0 = (x+1)y'' - (x+2)y' + y = e^x[(x+1)u'' + xu']$. So $w(x) = u'(x)$ satisfies a linear 1st order equation $(x+1)w' + xw = 0$. In the canonical form we have $w' + \frac{x}{x+1} w = 0$. So $\int \frac{x}{x+1} dx = \int (1 - \frac{1}{x+1})dx = x - \ln|x+1|$, the integrating factor is $\exp(x - \ln|x+1|) = \frac{1}{x+1} e^x$ and $(\frac{1}{x+1} e^x w)' = 0$. Take $w = (x+1)e^{-x}$. Then $u = \int w(x)\,dx = -(x+2)e^{-x} + C$ and $y_2(x) = e^x u(x) = -(x+2) + Ce^x$. A simplest 2nd solution is $y_2(x) = -e^x u(x) = (x+2)$ when $C = 0$. Substitution into the equation confirms the result. The general solution of the inhomogeneous equation is $y(x) = Ae^x + B(x+2) + x^2$.

----

4. Equations $0 = y^2 - x^2 = (y-x)(y+x) = f(x,y)$, $0 = x-1 = g(x,y)$ give 2 equilibrium points $(x_1 = 1, y_1 = 1)$, $(x_2 = 1, y_2 = -1)$, The Jacobian $J(x,y) = \begin{bmatrix} -2x & 2y \\ 1 & 0 \end{bmatrix}$. We list the points and their stability.

$J(1,1) = \begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1+\sqrt{3} > 0$, $\lambda_2 = -1-\sqrt{3} < 0$ and the point is unstable.

$J(1,-1) = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_\pm = -1 \pm i$,, $\operatorname{Re}\lambda_\pm < 0$ and the point is stable.

The condition $0 = f(x,y) = (y-x)(y+x)$ gives 2 nullclines $y = x$, $y = -x$ with the vectofield parallel to y-axis. The condition $0 = g(x,y) = x-1$ defines a vertical line $x = 1$ where the vectorfield is parallel to x-axis. Intersections of these nullclines give 2 equilibrium points.

----

5. Equations $\dot{x} = y = 0$, $\dot{y} = -4x - ay = 0$ have only one real solution (0,0). Take the ansatz $V = bx^2 + cy^2$ for Liapunov function.The directional derivative is $\dot{V} = \frac{d}{dt}(bx^2 + cy^2) = 2bxy + 2cy(-4x - ay) =$ $(2b - 8c)xy - 2acy^2 = -2acy^2 \le 0$ when $b = 4, c = 1$. So $\dot{V} = -12ay^6 \le 0$ is negative semidefinite for $a > 0$. Thus $V(x,y) = (4x^2 + y^2)$ is a suitable positively definite Liapunov function satisfying $V(0,0) = 0$ and having nonpositive directional derivative. It is asymptoticaly stable if (0,0) is the only solution in the set $M = \{(x,y) : 0 = \dot{V}(x,y) = -12ay^6\} = \{(x = 0, y)\}$. If a solution satisfies $y(t) = 0$ then $\dot{y} = -4x - ay = 0$ implies that also $x(t) = 0$. So only the constant solution $(x(t) = 0, y(t) = 0)$ belongs to M and by the Liapunov theorem the point $(0,0)$ is also asymptotically stable. The Jacobian $J(0,0) = \begin{bmatrix} 0 & 1 \\ -4 & -a \end{bmatrix}$ has eigenvalues

$\lambda_\pm = \frac{1}{2}[-a \pm \sqrt{a^2 - 16}]$ which are complex when $0 < a < 4$ and real with $\sqrt{a^2 - 16} < a$ when $a \ge 4$. In both cases $\operatorname{Re}\lambda_\pm < 0$. Thus the equilibrium (0,0) is asymptoticaly stable.

----

6. For proof see book p.724. The integral equations $y(x) + 2x - 1 = \int_0^x (y(s)^2 + 1)ds$ is equivalent to the differential equation $y' + 2 = y^2 + 1$ with the initial condition $y(0) = 1$. This is a separable equation. Rewrite it as $\frac{1}{(y-1)(y+1)} y' = 1$ when $y - 1 \ne 0$ and $y + 1 \ne 0$. $\int 1 dx = \int \frac{1}{(y-1)(y+1)} y' dx = \int \frac{1}{2}(\frac{1}{y-1} - \frac{1}{(y+1)}) dy = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|$ and $\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = x + C$. Then $\frac{y-1}{y+1} = \pm e^{2C} e^{2x} = De^{2x}$ with $\pm e^C = D \ne 0$ and the solution is $y = \frac{1+De^{2x}}{1-De^{2x}}$. The conditions $y - 1 \ne 0$ and $y + 1 \ne 0$ exclude two constant solutions $y = 1$ and $y = -1$. So in the general solution $y = \frac{1+De^{2x}}{1-De^{2x}}$ the constant $D \in \mathbf{R}$ is allowed to take an arbitrary real value; for $D = 0$, $y(x) = 1$. The solutions $y = -1$ is recovered when $D \to \infty$. The initial condition: $1 = y(0) = \frac{1+De^{2x}}{1-De^{2x}}$ gives $D = 0$ and $y(x) = 1$. Control: $lhs = 2x, rhs = \int_0^x (1+1)\,ds = 2x$.