Tekniska Högskolan i Linköping, Matematiska Institutionen
Stefan Rauch

Tentamen i Ordinära Differentialekvationer och Dynamiska System 2017-01-11 kl. 8 - 13
No aids allowed. **Kurskod: TATA71, Provkod: TEN**

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 points and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Find a general solution for the first order differential equation: $\frac{dy}{dx} = y + y^2$.

a) Through the method of separation of variables b) By solving it as a Bernoulli equation.
Show that it is the same general solution meaning that domains of the integration constant are the same.
c) Give a solution for the following initial conditions: (i) $y(0) = -1$, (ii) $y(-1) = 2$.

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2. Use the method of variation of parameters to find a general solution for the equations

$$\dot{x} = -3x + 2y + t, \quad \dot{y} = -3x + 4y + t$$

Control the solution.

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3. For a linear, inhomogeneous 2nd order differential equation: $x^2 y'' + p(x)y' + 6y = h(x)$, $x > 0$ there is known one solution $y_1(x) = x^2$ of the homogeneous equation and a particular solution $y_p(x) = \ln x$ of the inhomogeneous equation. Find the unknown coefficients $p(x), h(x)$ of the inhomogeneous equation. Give a general solution of this inhomogeneous equation. You may confirm your result by reducing the homogeneous equation to an equation with constant coefficient by making change of the independent variable $t = \ln x$ and by solving this homogeneous equation.

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4. Find all equilibrium points and decide their linear stability for the dynamical system

$$\dot{x} = 3x - x^2 - xy, \qquad \dot{y} = y - 3xy + y^2$$

Draw all nullclines and direction of the vectorfield on the nullclines.

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5. Formulate the Liapunov theorem on asymptotic stability of an equilibrium point of an autonomous dynamical system. Show by taking the ansatz $V = ax^{2m} + by^{2n}$, $m,n \in \mathbf{N}$ for a Liapunov function that (0,0) is an asymptotically stable equilibrium point for the system $\dot{x} = -2y^3$, $\dot{y} = x - 3y^3$ and therefore all solutions go to (0,0) when $t \to \infty$. Can you confirm this result by using the linear criterion of stability?

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6. Consider the linear inhomogeneous equation with constant coefficients of problem 2 denoted as:
$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ t \end{bmatrix} = Az(t) + f(t)$. Show that $Az(t) + f(t)$ satisfies the Lipschitz condition and therefore the initial value problem $\dot{z} = Az(t) + f(t)$, $z(t=0) = [x(0) = 0, y(0) = 0]$ has a unique solution. Find this solution.

TATA71 2017-01 1. To separate variables rewrite the equation as $\frac{1}{y(y+1)}y' = 1$ when $y+1 \neq 0$ and $y \neq 0$. By integrating both sides we get $\int 1\,dx = \int \frac{1}{y(y+1)} y'\,dx = \int (\frac{1}{y} - \frac{1}{(y+1)})dy$ and $\ln\left|\frac{y}{y+1}\right| = x + C$. Then $\frac{y}{y+1} = \pm e^C e^x = De^x$ with $\pm e^C = D \neq 0$ and the solution is $y = \frac{De^x}{1-De^x}$. The conditions $y+1 \neq 0$ and $y \neq 0$ exclude two constant solutions $y = -1$ and $y = 0$. So in the general solution $y = \frac{De^x}{1-De^x}$ the constant $D \in \mathbf{R}$ is allowed to take an arbitrary real value. The Bernoulli equation $\frac{dy}{dx} - y = y^2$ can be rewritten $y^{-2}\frac{dy}{dx} - y^{-1} = 1$, when $y \neq 0$. The substitution $z = y^{-1}$ gives the linear equation $z' + z = -1$ having an integrating factor e^x. Thus $(e^x z)' = -e^x$, $z = -1 + Ce^{-x}$ and $y = 1/(Ce^{-x} - 1)$ with $C \in \mathbf{R}$. By rewriting $y = 1/(Ce^{-x} - 1) = C^{-1}e^x/(1 - C^{-1}e^x)$ we obtain the same form of the solution when $C \neq 0$. $C = 0$ gives the constant solution $y(x) = -1$ and the limit $C \to \infty$ gives $y(x) = 0$. Thus we have the same set of solutions coming from both methods. (i) the solution $y(x) = -1$ satisfies $y(0) = -1$ and (ii) $2 = y(-1) = 1/(Ce - 1)$ gives $C = \frac{3}{2e}$ and the solution $y = 1/(\frac{3}{2}e^{-x-1} - 1)$

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2. By deriving $\ddot{x} = -3\dot{x} + 2(-3x + 4y + t) + 1 = \dot{x} + 6x - 2t + 1$ we get the inhomogeneous equation $\ddot{x} - \dot{x} - 6x = -2t + 1$. The the characteristic equation is $\lambda^2 - \lambda + 6 = (\lambda - 3)(\lambda + 2)$ and the general solution $x_h(t) = Ae^{3t} + Be^{-2t}$. The ansatz $x_p(t) = at + b$ gives a particular solution $x_p(t) = \frac{1}{3}t - \frac{2}{9}$ and the general solution $x_g(t) = Ae^{3t} + Be^{-2t} + \frac{1}{3}t - \frac{2}{9}$. From the first equation $y = \frac{1}{2}(\dot{x} + 3x - t) = 3Ae^{3t} + \frac{1}{2}Be^{-2t} - \frac{1}{6}$.
To use the variation of parameters formula denote $x_1(t) = e^{3t}, x_2(t) = e^{-2t}$ then
$W(t) = x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t) = -5e^t$ and with $h(t) = -2t + 1$
$x_p(t) = -x_1(t)\int x_2(t)\frac{h(t)}{W(t)}dt + x_2(t)\int x_1(t)\frac{h(t)}{W(t)}dt = \frac{1}{3}t - \frac{2}{9}$. It is the same particular solution as above.

The ansatz $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}$ gives $\lambda \mathbf{w} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = A\mathbf{w}$ with eigenvectors

$w_+ = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $w_- = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_+ = 3, \lambda_- = -2$. The general solution of the homogeneous system is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} + B\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}$.

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3. By substituting $y_1(x) = x^2$ into the homogeneous equation $0 = x^2 y_1'' + p(x)y_1' + 6y_1 = [y_1 = x^2] = 2x^2 + p(x)2x + 6x^2$ we get $p(x) = -4x$. From the equation $h(x) = x^2 y_p'' - 4xy_p' + 6y_p = -5 + 6\ln x$ and the equation reads: $x^2 y'' - 4xy' + 6y = -5 + 6\ln x$. For 2nd solution the ansatz $y_2(x) = z(x)x^2$ gives $0 = x^2 y_2'' - 4xy_2' + 6y_2 = [y_2 = z(x)x^2] = x^4 z''$ and $z(x) = Ax + B$. This gives $y_2(x) = x^3$ and the general solution is $y_{gen}(x) = Ay_1(x) + By_2(x) + y_p(x) = Ax^2 + Bx^3 + \ln x$.

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4. Equations $0 = 3x - x^2 - xy = x(3 - x - y) = f(x, y)$, $0 = y - 3xy + y^2 = y(1 - 3x + y) = g(x, y)$ give 4 equilibrium points $(x_1 = 0, y_1 = 0)$, $(x_2 = 0, y_2 = -1)$, $(x_3 = 3, y_3 = 0)$ and $(x_4 = 1, y_4 = 2)$. The Jacobian is $J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 3 - 2x - y & -x \\ -3y & 1 - 3x + 2y \end{bmatrix}$. We list the points and their stability.

$J(0,0)=\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1 > 0$, $\lambda_2 = 3 > 0$, unstable. $J(0,-1)=\begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$, $\lambda_1 = 4 > 0$, $\lambda_2 = -1 < 0$, unstable. $J(3,0)=\begin{bmatrix} -3 & -3 \\ 0 & -8 \end{bmatrix}$, $\lambda_1 = -3 < 0$, $\lambda_2 = -8 < 0$, stable. $J(1,2)=\begin{bmatrix} -1 & -1 \\ -6 & 2 \end{bmatrix}$, $0 = Det[J(1,2)-\lambda] = \lambda^2 - \lambda - 8$ gives the eigenvalues $\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{33}$ and the point (1,2) is unstable since $\lambda_+ = \frac{1}{2} + \frac{1}{2}\sqrt{33} > 0$. The condition $0 = f(x,y) = x(3-x-y)$ gives 2 nullclines $x = 0, y = 3-x$ with the vectofield parallel to y-axis. The condition $0 = g(x,y) = y(1-3x+y)$ defines 2 straight lines $y = 0, y = 3x-1$ where the vectorfield is parallel to x-axis. Intersections of these nullclines give 4 equilibrium points.

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5. Equations $\dot{x} = -2y^3 = 0$, $\dot{y} = x - 3y^3 = 0$ have only one real valued solution (0,0). The directional derivative $\dot{V} = \frac{d}{dt}(ax^{2m} + by^{2n}) = 2max^{2m-1}\dot{x} + 2nby^{2n-1}\dot{y} = 2max^{2m-1}(-2y^3) + 2nby^{2n-1}(x-3y^3) =$. $-4max^{2m-1}y^3 + 2nby^{2n-1}x - 6nby^{2n-1}y^3 = [m=1, n=2] = -4axy^3 + 4by^3x - 12by^3y^3$

For $m = 1, n = 2$ and $a = b = 1$, $\dot{V} = -12by^6 \le 0$ is negative semidefinite. The function $V = x^2 + y^4 \ge 0$ is a positive definite Liapunov function taking value $V(0,0) = 0$ only at the equilibrium point. By the Liapunov theorem (0,0) is stable.

For showing asymptotic stability study the set $M = \{(x,y) : 0 = \dot{V} = -12ay^6\} = \{(x, y = 0)\}$.

It consits of whole x-axis. If a solution satisfies $y(t) = 0$ then $0 = \dot{y} = x(t) - 3y(t)^3 = x(t)$ and therefore also $x(t) = 0$. So only the constant solution $(x(t) = 0, y(t) = 0)$ belongs to M and by the Liapunov theorem the point $(0,0)$ is also asymptotically stable.

The Jacobian $J(0,0) = \left.\begin{bmatrix} 0 & -6y^2 \\ 1 & -9y^2 \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, has a double zero eigenvalue and the linear criterion of stability is unconclusive.

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6. A differential equation $\dot{z} = F(t,z),\ z \in \mathbf{R}^n, F(t,z) \in \mathbf{R}^n$ satisfies the Lipschitz condition when there is a constant L such that $\|F(t,z) - F(t,w)\| \le L\|z - w\|$ for z, w belonging to a certain open convex region of $\mathbf{R}^n$ .Here $\|F(t,z) - F(t,w)\| = \|A(z-w)\| \le \|A\|\|z-w\|$ where for a constant matrix the Lipschitz constant $L = \|A\| = \max(\lambda_m)$ is equal to maximal value of one of the eigenvalues λ_m of the matrix A .

Here the matrix $\begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}$ has eigenvalues -2, 3 so $L = 3$. Since the function Ay + f (x) is continuous and satisfies the Lipschitz property the assumptions of the existence and uniqueness theorem are satisfied and the initial value problem has a unique solution that is defined for all values of x. The ansatz $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}$ gives $\lambda \mathbf{w} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = A\mathbf{w}$ with eigenvectors $w_+ = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $w_- = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ corresponding to the eigenvalues $\lambda_+ = 3, \lambda_- = -2$. The general solution of the homogeneous system is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} + B\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t}$. The IC $z(t=0) = [x(0) = 0, y(0) = 0]$ and the equations imply that also $\dot{x}(0) = 0$ and the solution is $x(t) = \frac{1}{45}e^{3t} + \frac{1}{5}e^{-2t} + \frac{1}{3}t - \frac{2}{9}$. From the equation $\dot{x} = -3x + 2y + t$ we gets $y(t) = \frac{1}{2}(\dot{x} + 3x - t) = \frac{1}{15}e^{3t} + \frac{1}{10}e^{-2t} - \frac{1}{6}$.