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Tentamen i Ordinära Differentialekvationer och Dynamiska System 2017-04-19 kl.14-19
No aids allowed. Kurskod: TATA71, Provkod: TEN

Solutions of all problems have to be complete and all arguments well motivated. When known theorems are used it has to be shown that the assumptions are fulfilled. Each problem is worth 3 points and 2 points are needed for having the problem approved. 3 passed problems and 8 points are needed for passing the examination.

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1. Consider the following logistic type equation: $\dot{N} = r[1-(\frac{N}{K})^2]N$ where $r, K > 0$.

a) Show by using the method of separation of variables that the solution of the initial value problem $N(t=0) = N_0$ is given by $N(t) = KN_0 \Big/ \sqrt{N_0^2 + (K^2 - N_0^2)e^{-2rt}}$

b) Find all equilibrium points and decide their stability.

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2. Find equilibrium points of the linear system of equations: $\dot{u} = 4u - 3w + 2$, $\dot{w} = 3u + 4w - 11$ and decide their stability. Confirm your result by calculating a general solution of this linear system of equations. Draw an approximate phase portrait for this system.

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3. A linear 2-nd order nonhomogeneous ODE $y''(x) + p(x)y'(x) + q(x)y(x) = h(x)$ has a general solution $y(x) = Ae^{-x} + Bxe^{x} - 1$. Find the equation and verify that the homogeneous solution and the particular solution satisfy the determined equation. Give a solution of IC: $y(0) = y'(0) = 0$.

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4. For the dynamical system: $\dot{x} = 7x - x^2 - 2xy$, $\dot{y} = 5y - y^2 - xy$

find all equilibrium points and decide their linear stability. Draw all nullclines and direction of the vectorfield on the nullclines.

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5. Formulate the Liapunov theorem on asymptotic stability of an equilibrium point of an autonomous dynamical system. Find, by taking the ansatz $V = ax^2 + by^2$, $a, b > 0$ for a Liapunov function, that the equilibrium point (0,0) is asymptotically stable for the system

$$\dot{x} = xy^2 - \tfrac{1}{2}x^3, \quad \dot{y} = \tfrac{1}{5}x^2 y - \tfrac{1}{2}y^3$$

and, therefore all solutions go to (0,0) when $t \to \infty$. Can you confirm this result by using the linear criterion of stability?

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6. Formulate the theorem about existence and uniqueness of solutions for the initial value problem(IVP):

$$\frac{dy(x)}{dx} = f(x,y),\ y(0) = b, \qquad \text{where } x, y, f(x,y) \in \mathbf{R}$$

Show that both functions $y_1(x) = x^3, y_2(x) = 0$ satisfy the same IVP: $\dfrac{dy(x)}{dx} = 3y(x)^{2/3}, y(0) = 0$.

Explain which assumption of the theorem is not satisfied.

Solutions Tentamen i ODE´s TATA71 2017-04-19

1. a) Separation of variables gives $rt + C = \int r dt = \int \frac{K^2}{N(K+N)(K-N)} dN = \int [\frac{1}{N} - \frac{1/2}{N+K} - \frac{1/2}{N-K}] dN = \frac{1}{2} \ln \left| \frac{N^2}{N^2-K^2} \right|$.

By solving for N we get $\frac{N^2}{N^2-K^2} = \pm e^C e^{2rt} = De^{2rt}$ where $0 \neq D \in R$ is an arbitrary nonzero constant.

The initial condition $N(t=0) = N_0$ gives $D = N_0^2 / (N_0^2 - K^2)$ and the required formula.

b) $0 = r[1-(\frac{N}{K})^2]N = f(N)$ gives the equilibrium points $N_1 = 0, N_{\pm} = \pm K$. $f'(N) = r - 3r\frac{N^2}{K^2}$, and $f'(N_1 = 0) = r > 0, f'(N_{\pm} = \pm K) = r - 3r = -2r < 0$. This means that $N_1 = 0$ is unstable and $N_{\pm} = \pm K$ are stable.

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2. Equations $0 = 4u - 3w + 2$, $0 = 3u + 4w - 11$ have a single solution $(u_0 = 1, w_0 = 2)$. The substitution $(u = x+1, w = y+2)$ turns the initial equations into a homogeneous system of linear equations $\dot{x} = 4x - 3y \quad \dot{y} = 3x + 4y$ having equilibrium at (0,0). We can solve it either by reducing it to a single 2-nd order equation or by using a vector ansatz for a solution. By deriving $\dot{x} = 4x - 3y$ we get $\ddot{x} = 4\dot{x} - 3\dot{y} = 4\dot{x} - 9x - 12y = 4\dot{x} - 9x - 4(4x - \dot{x}) = 8\dot{x} - 25x$ and $\ddot{x} - 8\dot{x} + 25x = 0$.

$0 = \ddot{x} - 8\dot{x} + 25x = [x = e^{\lambda t}] = e^{\lambda t}(\lambda^2 - 8\lambda + 25)$ and $\lambda_{\pm} = 4 \pm 3i$. The general solution is

$x(t) = e^{4t}(A\cos 3t + B\sin 3t)$ and $y = \frac{4}{3}x - \frac{1}{3}\dot{x} = e^{4t}(A\sin 3t - B\cos 3t)$. The ansatz $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} e^{\lambda t}$

gives $0 = Det\begin{bmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{bmatrix} = (4-\lambda)^2 + 9 = \lambda^2 - 8\lambda + 23$. The eigenvector associated with $\lambda_{-} = 4 - 3i$

is $\mathbf{w}_{-} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and the complex solution $\mathbf{x}(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(4-3i)t} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} e^{4t}(\cos 3t - i \sin 3t)$. This

gives two real-valued solutions $\mathbf{x}_1(t) = e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}$, $\mathbf{x}_2(t) = e^{4t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}$ and the same general solution

$\mathbf{x}_1(t) = e^{4t}(C\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + D\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix})$. The phase portrait has an equilibrium point (0,0) and trajectories are outgoing spirals emanating from (0,0). The phase portrait for $(u(t), w(t))$ has the same form with the equilibrium point moved to (1,2). As $\mathrm{Re}\,\lambda_{\pm} = \mathrm{Re}(4 \pm 3i) = 4 > 0$ the equilibrium (0,0) is unstable.

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3. By substituting $y_1(x) = e^{-x}, y_2(x) = xe^x$ into the homogeneous equation $y''(x) + py'(x) + qy(x) = 0$ we obtain 2 algebraic equations for two unknown functions: $p - q - 1 = 0, (1+x)p + xq + 2 + x = 0$. They have solutions $p(x) = -\frac{2}{1+2x}, q(x) = -\frac{3+2x}{1+2x}$. The particular solution $y_p(x) = -1$ substituted into the l.h.s of the equation gives $h(x) = \frac{3+2x}{1+2x}$. So the equation is $y''(x) - \frac{2}{1+2x}y'(x) - \frac{3+2x}{1+2x}y(x) = \frac{3+2x}{1+2x}$.

A solution of the IC is $y(x) = e^{-x} + xe^x - 1$.

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4. Equations $0 = 7x - x^2 - 2xy = f(x,y)$, $0 = 5y - y^2 - xy = g(x,y)$ give 4 equilibrium points $(x_1 = 0, y_1 = 0)$, $(x_2 = 0, y_2 = 5)$, $(x_3 = 7, y_3 = 0)$ and $(x_4 = 3, y_4 = 2)$. The Jacobian

is $J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 7-2x-2y & -2x \\ -y & 5-x-2y \end{bmatrix}$. We list the points and their stability.

$J(0,0)=\begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$ has eigenvalues $\lambda_1 = 7 > 0$, $\lambda_2 = 5 > 0$, unstable.

$J(0,5)=\begin{bmatrix} -3 & 0 \\ -5 & -5 \end{bmatrix}$ has eigenvalues $\lambda_1 = -3 > 0$, $\lambda_2 = -5 > 0$, stable.

$J(7,0)=\begin{bmatrix} -7 & -14 \\ 0 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -7 < 0$, $\lambda_2 = -2 < 0$, stable.

$J(3,2)=\begin{bmatrix} -3 & -6 \\ -2 & -2 \end{bmatrix}$, $\lambda_1 = 1 > 0$, $\lambda_2 = -6 > 0$, unstable.

The condition $0 = f(x,y) = x(7-x-2y)$ gives 2 nullclines $x = 0, y = \frac{1}{2}(7-x)$ with the vectofield parallel to y-axis. The condition $0 = g(x,y) = y(5-x-y)$ defines 2 straight lines $y = 0, y = 5-x$ where the vectorfield is parallel to x-axis. Intersections of these nullclines give 4 equilibrium points.

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5. Equations $\dot{x} = xy^2 - \frac{1}{2}x^3 = 0$, $\dot{y} = \frac{1}{5}x^2y - \frac{1}{2}y^3 = 0$ have only one real valued solution (0,0).
The directional derivative $\dot{V} = \frac{d}{dt}(ax^2+by^2) = 2ax^2(y^2 - \frac{1}{2}x^2) + 2by^2(\frac{1}{5}x^2 - \frac{1}{2}y^2) =$
$-ax^4 + (2a + \frac{2}{5}b)x^2y^2 - by^4$. A choice $a = 1, b = 5$ gives
$\dot{V}\big|_{a=1,b=5} = -ax^4 + (2a+\frac{2}{5}b)x^2y^2 - by^4\big|_{a=1,b=5} = -x^4 + 4x^2y^2 - 5y^4 = (-x^4 + 4x^2y^2 - 4y^4) - y^4 =$
$= -(x^4 - 2y^2)^2 - y^4 < 0$ being negative definite. The function $V = x^2 + 5y^4 \geq 0$ is a positive definite Liapunov function taking value $V(0,0) = 0$ only at the equilibrium point. By the Liapunov theorem (0,0) is asymptotically stable. The Jacobian
$J(0,0)=\begin{bmatrix} y^2 - \frac{3}{2}x^2 & 2xy \\ \frac{2}{5}xy & \frac{1}{5}x^2 - \frac{3}{2}y^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, has only zero eigenvalues and the linear criterion is unconclusive.

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6. For theorem see the textbook.
By separating variables we get $\frac{1}{3}y^{-2/3}y' = 1$ and the solution is $y_1(x) = (x+C)^3$. From the IC $0 = y_1(0) = (C)^3$ the constant is $C = 0$. So $y(x) = x^3$ is a solution and obviously $y(x) = 0$ satisfies the equation. But the Lipschitz condition guaranteeing uniqueness of solutions is not satisfied here. Here $f(y) = 3y^{2/3}$.By the mean value theorem there is a value $\xi(y)$ between 0 and y such that $|f(y) - f(0)| = \left|\frac{\partial f(\xi(y))}{\partial y}\right| |y - 0|$. But $\left|\frac{\partial f(\xi(y))}{\partial y}\right| = \left|2\xi(y)^{-1/3}\right| \xrightarrow{y \to 0} \infty$ is unbounded as $y \to 0$, since $\xi(y)$ stays between 0 and y. So there is no Lipshitz constant L so that $|f(y) - f(0)| \leq L|y - 0|$ in a certain neighborhood of $y = 0$.