Linköpings universitet
Matematiska institutionen
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Kurskod: TATA71
Provkod: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2018-01-13 kl. 8.00–13.00

No aids allowed. You may write your answers in English or Swedish (or both).

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Draw the phase portrait for the equation $\dot{x} = (x-1)(x-3)$, and compute the solution $x(t)$ which satisfies the initial condition $x(0) = 2$.

2. Compute the solution $(x_1(t), x_2(t))$ of the system $\dot{x}_1 = 5x_1 - 15x_2$, $\dot{x}_2 = 3x_1 - 7x_2$ with initial values $x_1(0) = c_1$, $x_2(0) = c_2$.

   What type of equilibrium is the origin? In the x_1x_2-plane, draw the null-clines with arrows indicating the direction of the vector field, and also carefully draw the solution curve passing through the point $(1,0)$.

3. Show that the origin is an asymptotically stable equilibrium for the system $\dot{x} = y - x^3 + \frac{1}{9}x^5$, $\dot{y} = -4x - y^3 + y^5$, and determine a domain of stability.

   (Hint: look for a strong Liapunov function of the form $V(x,y) = x^2 + \alpha y^2$.)

4. Find all equilibrium points for the system $\dot{x} = (x+1)y$, $\dot{y} = x^2 - 2x - y$, and use linearization to determine their type. Draw figures indicating what the local phase portrait near each equilibrium looks like (with the correct principal directions, in case the eigenvalues are real).

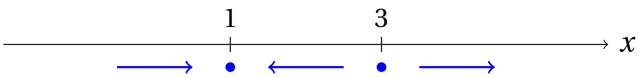
5. (a) For the system in problem 4, show that the line $x + y = 2$ is invariant. (That is, any solution starting on that line stays on that line.)
   (Hint: What is $\frac{d}{dt}(x+y-2)$?)

   (b) Use this information, together with problem 4, to draw the global phase portrait.

6. The function $x(t) = e^t$ satisfies the homogeneous ODE $t\ddot{x} - (1+t)\dot{x} + x = 0$. Use this to find the general solution of $t\ddot{x} - (1+t)\dot{x} + x = t^2 e^{2t}$ (for $t > 0$).

Solutions for TATA71 2018-01-13

1. Phase portrait:



   The solution satisfying $x(0) = 2$ is not one of the constant solutions $x = 1$ or $x = 3$, so we can find it using separation of variables:

$$\int_{x(0)}^{x(t)} \frac{d\xi}{(\xi-1)(\xi-3)} = \int_0^t d\tau \quad \Longleftrightarrow \quad \left[\frac{1}{2} \ln \left| \frac{\xi-3}{\xi-1} \right| \right]_2^{x(t)} = t.$$

   Since $1 < x(t) < 3$ for all t (according to the phase portrait), we can get rid of the absolute value signs like this:

$$\ln \frac{3 - x(t)}{x(t) - 1} - \ln 1 = 2t.$$

   Answer: $x(t) = \dfrac{e^{2t}+3}{e^{2t}+1}$.

2. The system is

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad A = \begin{pmatrix} 5 & -15 \\ 3 & -7 \end{pmatrix},$$

   and the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0).$$

   The eigenvalues of A are $a \pm ib = -1 \pm 3i$, and a complex eigenvector corresponding to the eigenvalue $-1+3i$ is

$$\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   We know that taking the imaginary part and the real part as columns in

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

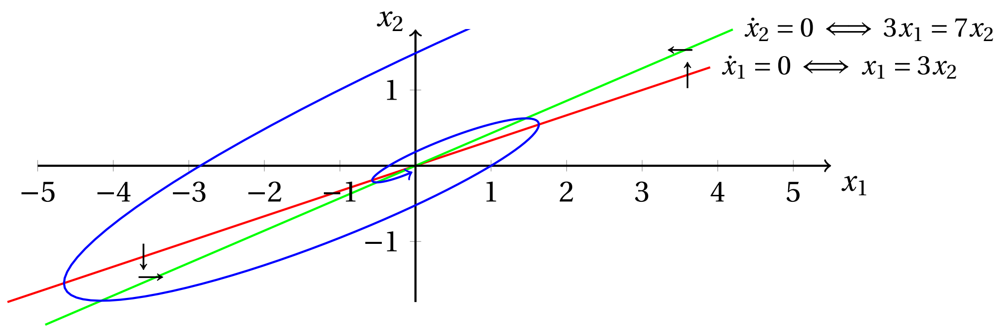
   gives the transformation to normal form,

$$J = M^{-1}AM = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix}, \quad A = MJM^{-1}.$$

Hence the solution is

$$\begin{aligned}
\mathbf{x}(t) = e^{At}\mathbf{x}(0) &= Me^{Jt}M^{-1}\mathbf{x}(0) \\
&= e^{-t}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos 3t & -\sin 3t\\ \sin 3t & \cos 3t\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} \\
&= e^{-t}\begin{pmatrix}\cos 3t + 2\sin 3t & -5\sin 3t\\ \sin 3t & \cos 3t - 2\sin 3t\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}.
\end{aligned}$$

The origin is a **stable focus**, with orbits spiralling counterclockwise inwards.



3. With $V(x,y) = x^2 + \alpha y^2$, we compute $\dot{V} = 2x\dot{x} + 2\alpha y\dot{y} = 2x(y - x^3 + \frac{1}{9}x^5) + 2\alpha y(-4x - y^3 + y^5) = 2(1-4\alpha)xy - 2x^4 + \frac{2}{9}x^6 - 2\alpha y^4 + 2\alpha y^6$. With $\alpha = 1/4$, the xy term vanishes, and we get

$$V = x^2 + \tfrac{1}{4}y^2, \qquad \dot{V} = -2x^4(1 - \tfrac{1}{9}x^2) - \tfrac{1}{2}y^4(1-y^2).$$

Then V is positive definite, and $\dot{V}$ is negative definite in the open set $\Omega = \{(x,y) \in \mathbf{R}^2 : |x| < 3, |y| < 1\}$, so by Liapunov's theorem the origin is asymptotically stable.

If B is an origin-centered closed disc which fits inside Ω (i.e., with radius $r < 1$), then the smallest value of V on the boundary of B is $V(0, \pm r) = \frac{1}{4}r^2$, and the set $\{(x,y) \in B : V(x,y) < \frac{1}{4}r^2\}$ is a domain of stability. Since this is true for any $r < 1$, the ellipse

$$\{(x,y) \in B : V(x,y) < \tfrac{1}{4}\} = \{(x,y) \in \mathbf{R}^2 : 4x^2 + y^2 < 1\}$$

is a domain of stability.

4. The equilibrium points are given by $\dot{x} = (x+1)y = 0$ and $\dot{y} = x^2 - 2x - y = 0$. The first equation gives $x = -1$ or $y = 0$, which together with the second equation gives $y = 3$ or $x \in \{0, 2\}$, respectively. So the equilibria are $(-1, 3)$, $(0, 0)$ and $(2, 0)$.

   The Jacobian matrix of the system is

$$A(x, y) = \begin{pmatrix} y & x+1 \\ 2x-2 & -1 \end{pmatrix},$$

   and in particular

$$A(-1,3) = \begin{pmatrix} 3 & 0 \\ -4 & -1 \end{pmatrix}, \qquad A(0,0) = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \qquad A(2,0) = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}.$$

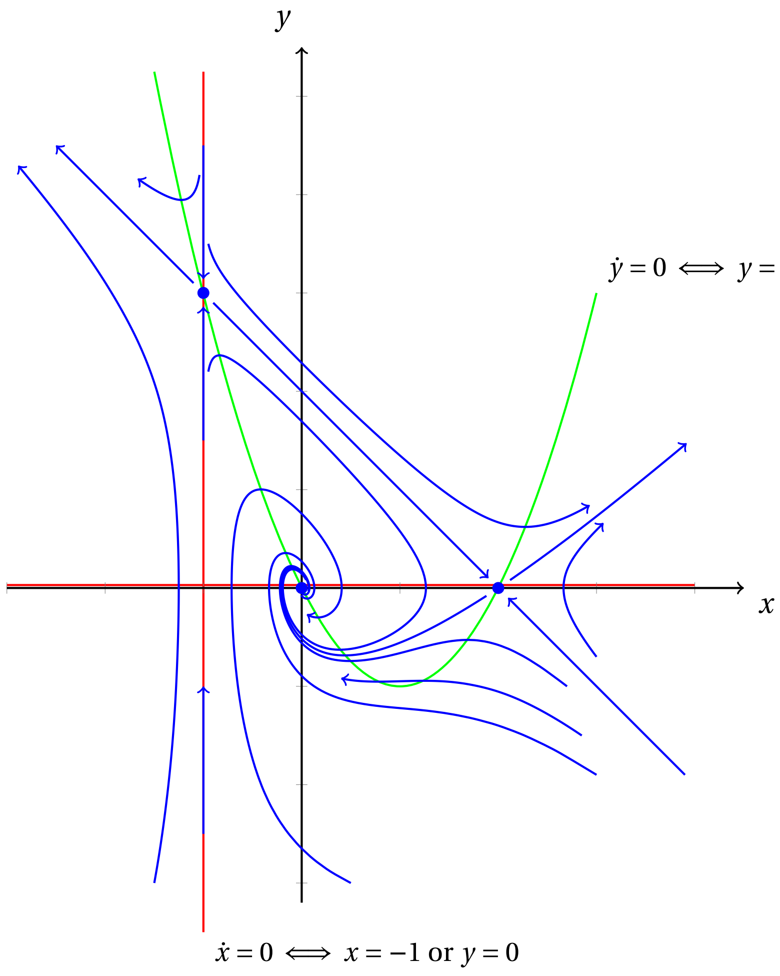
   This shows that $(-1, 3)$ is a saddle, with principal directions given by the eigenvectors $(-1, 1)^T$ for the positive eigenvalue $\lambda = 3$ and $(0, 1)^T$ for the negative eigenvalue $\lambda = -1$. Likewise, $(2, 0)$ is a saddle, with principal directions $(3, 2)^T$ for $\lambda = 2$ and $(-1, 1)^T$ for $\lambda = -3$. And $(0, 0)$ is a stable focus, since $\mathrm{tr}(A) = -1 < 0$ and $\det(A) = 2 > \mathrm{tr}(A)^2/4$.

   For illustrations of the local phase portraits, zoom in near the equilibrium points in the global phase portrait in problem 5 below.

5. (a) If $(x(t), y(t))$ is a solution of the system, then $\frac{d}{dt}(x+y-2) = \dot{x}+\dot{y}-0 = (x+1)y+(x^2-2x-y) = x^2+xy-2x = x(x+y-2)$, so the quantity $u(t) = x(t)+y(t)-2$ will stay zero if it equals zero initially. Thus, solutions starting on the line $x+y-2=0$ stay there.

[Actually the equation $\dot{u} = xu$ tells us even more, namely that $u(t)$ will increase in absolute value if $x(t) > 0$; in other words, as long as a solution curve lies in the right half plane, it will keep increasing its distance to the line $x+y=2$. Similarly, all solutions in the left half plane $x<0$ move closer and closer to the line $x+y=2$.]

(b) Phase portrait:



6. First we use reduction of order to find another, linearly independent, solution of the homogeneous equation. With $x(t) = e^t Y(t)$ and $y(t) = \dot{Y}(t)$ we get

$$0 = t\ddot{x} - (1+t)\dot{x} + x = e^t\Big(t(\ddot{Y} + 2\dot{Y} + Y) - (t+1)(\dot{Y} + Y) + Y\Big) = e^t\big(t\dot{y} + (t-1)y\big).$$

For $t > 0$ this is equivalent to $\dot{y} + (1 - t^{-1})y = 0$, which we multiply by the integrating factor $\exp(t - \ln t) = e^t/t$ to obtain $\frac{d}{dt}(y(t)e^t/t) = 0$. Thus $y(t) = Cte^{-t}$, which gives $Y(t) = \int y(t)\,dt = -C(t+1)e^{-t} + D$. Since we are only looking for one solution, we take $C = -1$ and $D = 0$, so $Y(t) = (t+1)e^{-t}$, and consequently $x(t) = e^t Y(t) = t+1$. Now we know that the general solution of the homogeneous equation is $x(t) = C_1(t+1) + C_2 e^t$.

Next, rewrite the inhomogeneous equation $t\ddot{x} - (1+t)\dot{x} + x = t^2 e^{2t}$, first as $\ddot{x} - (1+t^{-1})\dot{x} + t^{-1}x = te^{2t}$, and then (with $x_1 = x$ and $x_2 = \dot{x}$) as the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -t^{-1} & 1 + t^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ te^{2t} \end{pmatrix}.$$

A fundamental matrix for the homogeneous system is

$$\Phi(t) = \begin{pmatrix} t+1 & e^t \\ \frac{d}{dt}(t+1) & \frac{d}{dt}e^t \end{pmatrix} = \begin{pmatrix} t+1 & e^t \\ 1 & e^t \end{pmatrix},$$

and the "variation of constants" substitution $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$ leads in the usual way to

$$\dot{\mathbf{y}}(t) = \Phi(t)^{-1} \begin{pmatrix} 0 \\ te^{2t} \end{pmatrix} = \frac{1}{te^t} \begin{pmatrix} e^t & -e^t \\ -1 & t+1 \end{pmatrix} \begin{pmatrix} 0 \\ te^{2t} \end{pmatrix} = \begin{pmatrix} -e^{2t} \\ (t+1)e^t \end{pmatrix}.$$

Integrating (and omitting constants of integration, since we are only looking for one particular solution), we find

$$\mathbf{y}(t) = \begin{pmatrix} -\frac{1}{2}e^{2t} \\ te^t \end{pmatrix}$$

and finally the particular solution

$$x(t) = x_1(t) = \Phi(t)_{\text{row }1}\,\mathbf{y}(t) = (t+1, e^t) \begin{pmatrix} -\frac{1}{2}e^{2t} \\ te^t \end{pmatrix} = \tfrac{1}{2}(t-1)\,e^{2t}.$$

Answer: $x(t) = c_1(t+1) + c_2 e^t + \frac{1}{2}(t-1)\,e^{2t}$.