Linköpings universitet
Matematiska institutionen
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Kurskod: TATA71
Provkod: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2018-04-04 kl. 14.00–19.00

No aids allowed. You may write your answers in English or Swedish (or both).

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Compute the general solution of the linear system
$$\dot{x} = 2y, \qquad \dot{y} = x - y$$
and sketch the phase portrait. Draw in particular, as precisely as you can, the trajectory through the point $(2,1)$ and the one through $(0,-1)$.

2. Transform the system
$$\dot{x} = x(4 - x^2 - y^2) - 10y, \qquad \dot{y} = y(4 - x^2 - y^2) + 10x$$
into polar coordinates. Sketch the phase portrait. Are there any limit cycles? If so, investigate their stability.

3. Use linearization to classify all equilibrium points of the system
$$\dot{x} = x^2 - xy, \qquad \dot{y} = 2 - x^3 - y.$$
Sketch the phase portrait.

4. Write the second-order ODE $\ddot{x} = x^3 - x$ as a first-order system by letting $y = \dot{x}$. Determine a constant of motion $F(x,y)$ for the system, and sketch the phase portrait. For which initial data $x(0) = a$, $\dot{x}(0) = b$ is the solution of the original ODE periodic?

5. Show that the origin is a globally asymptotically stable equilibrium for the system
$$\dot{x} = -2x + 3y - y^3, \qquad \dot{y} = -x + y - y^3.$$
(Hint: look for a strong Liapunov function $V(x,y) = x^2 + axy + by^2$.)

6. Derive a formula for the solution $x(t)$ of the initial value problem
$$(2t^2+1)\ddot{x}(t) - 4t\dot{x}(t) + 4x(t) = f(t), \qquad x(0) = a, \qquad \dot{x}(0) = b,$$
in terms of an integral involving the function f. (Hint: $x(t) = 2t^2 - 1$ is a solution of the homogeneous equation.)

Solutions for TATA71 2018-04-04

1. From the eigenvalues and eigenvectors of the system matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

we obtain the general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_1 e^{\lambda_2 t}\mathbf{v}_2 = C_1 e^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since there is one positive and one negative eigenvalue, the origin is a saddle point. The trajectory through $(2,1)$ is a half-line from the origin. The trajectory through $(-1,0)$ is a curve with the lines $y = x/2$ and $y = -x$ as asymptotes; it has the direction $(-2,1)^T$ when it passes the point $(0,-1)$, and its highest point is when it crosses the nullcline $y = x$.

For a sketch of the phase portrait, type "streamplot {2y,x-y}" into Wolfram Alpha (or simply click on the link).

2. The usual formulas $r\dot{r} = x\dot{x} + y\dot{y}$ and $r^2\dot{\theta} = -y\dot{x} + x\dot{y}$ give the decoupled system

$$\dot{r} = r(4 - r^2), \qquad \dot{\theta} = 10.$$

For $r \geq 0$, the one-dimensional phase portrait for the r equation is

$$0 \longrightarrow 2 \longleftarrow$$

so $r = 2$ is a stable equilibrium, which corresponds to the origin-centered circle of radius 2 being a stable limit cycle for the original system. The other trajectories spiral towards this circle, counter-clockwise since $\dot{\theta} > 0$. There is also an equilibrium at $(0,0)$, which is an unstable spiral, which we can see from the above, or from the fact that the linearized system

$$\dot{x} = 4x - 10y, \qquad \dot{y} = 10x + 4y$$

has the eigenvalues $4 \pm 10i$.

[Link to phase portrait.]

3. We have $\dot{x} = 0$ iff $x = 0$ or $y = x$. Inserting this into the equation $\dot{y} = 0$, we find the equilibrium points $(x,y) = (0,2)$ and $(1,1)$. The Jacobian matrix is

$$A(x,y) = \begin{pmatrix} 2x - y & -x \\ -3x^2 & -1 \end{pmatrix}, \quad A(0,2) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(1,1) = \begin{pmatrix} 1 & -1 \\ -3 & -1 \end{pmatrix}.$$

Thus, $(0,2)$ is a stable node since $A(0,2)$ obviously has the negative eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$, with the principal directions $(1,0)^T$ and $(0,1)^T$. And $(1,1)$ is a saddle (hence unstable) since $\det A(1,1) < 0$; the principal directions are $(1,3)^T$ (for $\lambda_1 = -2$) and $(-1,1)^T$ (for $\lambda_2 = 2$).

[Link to phase portrait.]

4. The second-order system is

$$\dot{x} = y, \qquad \dot{y} = x^3 - x.$$

This gives $dy/dx = \dot{y}/\dot{x} = (x^3 - x)/y$, so that $\int y\,dy = \int (x^3 - x)\,dx$, i.e., $\frac{1}{2}y^2 = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$. So we find the constant of motion

$$F(x,y) = x^2 + y^2 - \tfrac{1}{2}x^4.$$

(In the derivation, we excluded the case $y = 0$, but direct computation shows that $\dot{F} = 0$ always, so this is nothing to worry about.)

The linearization at the origin is $\dot{x} = y$, $\dot{y} = -x$, hence a centre, which gives no information about the nonlinear system, but since we have the constant of motion F we can say that $(0,0)$ is actually a nonlinear centre surrounded by closed curves (level curves of F). We have $F(x,y) \approx x^2 + y^2$ close to the origin, so the level curves there are approximately circles. The other equilibria are $(\pm 1, 0)$, and they are connected by the level curve $F = 1/4$. Inside this curve, we get periodic solutions, outside not. In formulas: the solution is periodic iff $a^2 + b^2 - \frac{1}{2}b^4 < \frac{1}{4}$ and $|a| < 1$.

[Link to phase portrait.]

5. With $V(x,y) = x^2 + axy + by^2$, we compute

$$\begin{aligned}
\dot{V} &= V_x\dot{x} + V_y\dot{y} \\
&= (2x + ay)(-2x + 3y - y^3) + (ax + 2by)(-x + y - y^3) \\
&= -(a+4)x^2 + (6 - a - 2b)xy + (3a + 2b)y^2 - (a + 2b)y^4 - (2 + a)xy^3.
\end{aligned}$$

The term xy^3 seems difficult to control, so we eliminate it by choosing $a = -2$, leaving

$$V = x^2 - 2xy + by^2 = (x - y)^2 + (b - 1)y^2$$

and

$$\dot{V} = -2x^2 + (8 - 2b)xy + (2b - 6)y^2 - (2b - 2)y^4.$$

Now V is positive definite iff $b-1>0$, and we also need $2b-6<0$ if there's going to be any chance for $\dot V$ to be negative definite. So we must have $1<b<3$. Trying $b=2$ gives

$$V=(x-y)^2+y^2, \qquad \dot V=-2(x-y)^2-2y^4.$$

It worked! We see that V is positive definite and $\dot V$ is negative definite, and moreover $V(x,y)\to\infty$ as $\sqrt{x^2+y^2}\to\infty$, so $(0,0)$ is globally asymptotically stable.

6. We were given one solution $x_1(t)=2t^2-1$ of the homogeneous equation, and a linearly independent solution $x_2(t)=t$ can be found by reduction of order (or by inspection). Now we can use "variation of constants". The corresponding first-order system

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\begin{pmatrix}0 & 1\\ -4/(1+2t^2) & 4t/(1+2t^2)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\ f(t)/(1+2t^2)\end{pmatrix}$$

has the fundamental matrix

$$\Phi(t)=\begin{pmatrix}-x_1 & x_2\\ -\dot x_1 & \dot x_2\end{pmatrix}=\begin{pmatrix}1-2t^2 & t\\ -4t & 1\end{pmatrix},$$

where (for convenience) the signs are chosen such that $\Phi(0)=I$. The new unknowns u and v defined by

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\Phi(t)\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}$$

then satisfy

$$\begin{pmatrix}\dot u\\ \dot v\end{pmatrix}=\Phi(t)^{-1}\begin{pmatrix}0\\ f(t)/(1+2t^2)\end{pmatrix}=\frac{1}{1+2t^2}\begin{pmatrix}1 & -t\\ 4t & 1-2t^2\end{pmatrix}\begin{pmatrix}0\\ f(t)/(1+2t^2)\end{pmatrix}=\frac{f(t)}{(1+2t^2)^2}\begin{pmatrix}-t\\ 1-2t^2\end{pmatrix}$$

and

$$u(0)=x(0)=a, \qquad v(0)=y(0)=\dot x(0)=b,$$

so that

$$u(t)=a+\int_0^t \frac{-s f(s)}{(1+2s^2)^2}\,ds, \qquad v(t)=b+\int_0^t \frac{(1-2s^2)f(s)}{(1+2s^2)^2}\,ds.$$

Finally, we get the answer from $x(t)=\Phi_{11}(t)u(t)+\Phi_{12}(t)u(t)$:

$$x(t)=a(1-2t^2)+bt+\int_0^t \frac{\big(-(1-2t^2)s+t(1-2s^2)\big)f(s)}{(1+2s^2)^2}\,ds.$$