Linköpings universitet
Matematiska institutionen
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Kurskod: TATA71
Provkod: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2019-01-18 kl. 8.00–13.00

No aids allowed. You may write your answers in English or Swedish (or both).
Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.
Solutions will be posted on the course webpage afterwards. Good luck!

1. Compute the general solution of the linear system
$$\dot{x} = 2y, \qquad \dot{y} = x - y$$
and sketch the phase portrait. Draw in particular, as precisely as you can, the trajectory through the point $(2,1)$ and the one through $(0,-1)$.

2. Transform the system
$$\dot{x} = x(4 - x^2 - y^2) - 10y, \qquad \dot{y} = y(4 - x^2 - y^2) + 10x$$
into polar coordinates, and use this to sketch the phase portrait. Are there any limit cycles? If so, investigate their stability.

3. Use linearization to classify all equilibrium points of the system
$$\dot{x} = x^2 - xy, \qquad \dot{y} = 2 - x^3 - y.$$
Sketch the phase portrait.

4. Write the second-order ODE $\ddot{x} = x^3 - x$ as a first-order system by letting $y = \dot{x}$. Determine a constant of motion $F(x,y)$ for the system, and sketch the phase portrait. For which initial data $x(0) = a$, $\dot{x}(0) = b$ is the solution of the original ODE periodic?

5. Show that the origin is a globally asymptotically stable equilibrium for the system
$$\dot{x} = -2x + 3y - y^3, \qquad \dot{y} = -x + y - y^3.$$
(Hint: look for a strong Liapunov function $V(x,y) = x^2 + axy + 2y^2$.)

6. Derive a formula for the solution $x(t)$ of the initial value problem
$$(2t^2+1)\ddot{x}(t) - 4t\dot{x}(t) + 4x(t) = f(t), \qquad x(0) = a, \qquad \dot{x}(0) = b,$$
in terms of an integral involving the function f.
(Hint: $x(t) = 1 - 2t^2$ and $x(t) = t$ satisfy the homogeneous equation.)

Solutions for TATA71 2019-01-18

1. From the eigenvalues and eigenvectors of the system matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

we obtain the general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_1 e^{\lambda_2 t}\mathbf{v}_2 = C_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Since there is one positive and one negative eigenvalue, the origin is a saddle point. The trajectory through $(2,1)$ is a half-line from the origin. The trajectory through $(-1,0)$ is a curve with the lines $y = x/2$ and $y = -x$ as asymptotes; it has the direction $(-2,1)^T$ when it passes the point $(0,-1)$, and its highest point is when it crosses the nullcline $y = x$.

For a sketch of the phase portrait, type "streamplot {2y,x-y}" into Wolfram Alpha (or simply click on the link).

2. The usual formulas $r\dot{r} = x\dot{x} + y\dot{y}$ and $r^2\dot{\theta} = -y\dot{x} + x\dot{y}$ give the decoupled system

$$\dot{r} = r(4 - r^2), \qquad \dot{\theta} = 10.$$

For $r \geq 0$, the one-dimensional phase portrait for the r equation is

$$0 \longrightarrow 2 \longleftarrow$$

so $r = 2$ is a stable equilibrium, which corresponds to the origin-centered circle of radius 2 being a stable limit cycle for the original system. The other trajectories spiral towards this circle, counter-clockwise since $\dot{\theta} > 0$. There is also an equilibrium at $(0,0)$, which is an unstable spiral, which we can see from the above, or from the fact that the linearized system

$$\dot{x} = 4x - 10y, \qquad \dot{y} = 10x + 4y$$

has the eigenvalues $4 \pm 10i$.

(I don't expect you to sketch the nullclines for this system; the equations are a bit too complicated for doing that by hand.)

[Link to phase portrait.]

3. We have $\dot{x} = 0$ iff $x = 0$ or $y = x$. Inserting this into the equation $\dot{y} = 0$, we find the equilibrium points $(x, y) = (0,2)$ and $(1,1)$. The Jacobian matrix is

$$A(x,y) = \begin{pmatrix} 2x - y & -x \\ -3x^2 & -1 \end{pmatrix}, \quad A(0,2) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(1,1) = \begin{pmatrix} 1 & -1 \\ -3 & -1 \end{pmatrix}.$$

Thus, $(0,2)$ is a stable node since $A(0,2)$ obviously has the negative eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$, with the principal directions $(1,0)^T$ and $(0,1)^T$. And $(1,1)$ is a saddle (hence unstable) since $\det A(1,1) < 0$; the principal directions are $(1,3)^T$ (for $\lambda_1 = -2$) and $(-1,1)^T$ (for $\lambda_2 = 2$).

[Link to phase portrait.]

4. The first-order system is

$$\dot{x} = y, \qquad \dot{y} = x^3 - x.$$

This gives $dy/dx = \dot{y}/\dot{x} = (x^3 - x)/y$, so that $\int y\,dy = \int (x^3 - x)\,dx$, i.e., $\frac{1}{2}y^2 = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$. So we find the constant of motion

$$F(x,y) = x^2 + y^2 - \tfrac{1}{2}x^4.$$

(In the derivation, we excluded the case $y = 0$, but direct computation shows that $\dot{F} = 0$ always, so this is nothing to worry about.)

The linearization at the origin is $\dot{x} = y$, $\dot{y} = -x$, hence a centre, which gives no information about the nonlinear system, but since we have the constant of motion F we can say that $(0,0)$ is actually a nonlinear centre surrounded by closed curves (level curves of F). We have $F(x,y) \approx x^2 + y^2$ close to the origin, so the level curves there are approximately circles. The other equilibria are $(\pm 1, 0)$, and they are connected by the level curve $F = 1/2$, which happens to be a union of two parabolas:

$$F(x,y) = x^2 + y^2 - \tfrac{1}{2}x^4 = \tfrac{1}{2} \quad \Longleftrightarrow \quad y^2 = \tfrac{1}{2}(x^2 - 1)^2 \quad \Longleftrightarrow \quad y = \pm \frac{x^2 - 1}{\sqrt{2}}.$$

Inside the bounded region formed by these parabolas, we get periodic solutions, outside not. In formulas: the solution is periodic iff $a^2 + b^2 - \frac{1}{2}b^4 < \frac{1}{2}$ and $|a| < 1$. That is, iff $-1 < a < 1$ and $a^2 - 1 < \sqrt{2}\,b < 1 - a^2$.

Links to some level curves: [$F = \frac{1}{4}$], [$F = \frac{1}{2}$], [$F = 1$]. [Link to phase portrait.]

5. With $V(x,y) = x^2 + axy + 2y^2$, we compute

$$\begin{aligned} \dot{V} &= V_x \dot{x} + V_y \dot{y} \\ &= (2x + ay)(-2x + 3y - y^3) + (ax + 4y)(-x + y - y^3) \\ &= -(a+4)x^2 + (2-a)xy + (3a+4)y^2 - (a+4)y^4 - (2+a)xy^3. \end{aligned}$$

The term xy^3 seems difficult to control, so we eliminate it by choosing $a=-2$, leaving

$$V = x^2 - 2xy + 2y^2 = (x-y)^2 + y^2,$$

which is positive definite, and

$$\dot{V} = -2x^2 + 4xy - 2y^2 - 2y^4 = -2(x-y)^2 - 2y^4,$$

which is negative definite. Thus V is a strong Liapunov function, and moreover $V(x,y) \to \infty$ as $\sqrt{x^2+y^2} \to \infty$, so $(0,0)$ is globally asymptotically stable.

6. We were given two linearly independent solutions of the homogeneous equation, $x_1(t) = 1-2t^2$ and $x_2(t) = t$, so the corresponding first-order system (with $y = \dot{x}$),

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4/(1+2t^2) & 4t/(1+2t^2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t)/(1+2t^2) \end{pmatrix},$$

has the fundamental matrix

$$\Phi(t) = \begin{pmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1-2t^2 & t \\ -4t & 1 \end{pmatrix}.$$

Now we can use "variation of constants": the usual calculation shows that the new unknowns u and v defined by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

satisfy

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \Phi(t)^{-1} \begin{pmatrix} 0 \\ \frac{f(t)}{1+2t^2} \end{pmatrix} = \frac{1}{1+2t^2} \begin{pmatrix} 1 & -t \\ 4t & 1-2t^2 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{f(t)}{1+2t^2} \end{pmatrix} = \frac{f(t)}{(1+2t^2)^2} \begin{pmatrix} -t \\ 1-2t^2 \end{pmatrix}$$

and (since $\Phi(0) = I$)

$$u(0) = x(0) = a, \qquad v(0) = y(0) = \dot{x}(0) = b,$$

so that

$$u(t) = a + \int_0^t \frac{-s\,f(s)}{(1+2s^2)^2}\,ds, \qquad v(t) = b + \int_0^t \frac{(1-2s^2)\,f(s)}{(1+2s^2)^2}\,ds.$$

Finally, we get the answer from $x(t) = \Phi_{11}(t)u(t) + \Phi_{12}(t)v(t)$:

$$x(t) = a(1-2t^2) + bt + \int_0^t \frac{\big((2t^2-1)s + t(1-2s^2)\big)\,f(s)}{(1+2s^2)^2}\,ds.$$