Linköpings universitet
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Kurskod: TATA71
Provkod: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2019-04-24 kl. 14.00–19.00

No aids allowed. You may write your answers in English or Swedish (or both).

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Draw the phase portrait for the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \qquad r > 0, \qquad K > 0,$$

and derive the exact formula for the general solution in terms of $x(0) = x_0$.

2. Sketch the phase portrait for the linear system

$$\dot{x} = -5y/4, \qquad \dot{y} = 2x - y.$$

Determine an explicit formula for the solution $(x(t), y(t))$ satisfying

$$x(0) = 1, \qquad y(0) = 0,$$

and draw that solution curve for $0 \le t \le 2\pi$ in the phase portrait.

3. Sketch the phase portrait for the system

$$\dot{x} = y - x^2, \qquad \dot{y} = x^2 + x - 2,$$

and use linearization to classify the equilibrium points.

4. State and prove the trace–determinant criterion for stability of a simple 2×2 linear system $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^2$, $\det(A) \neq 0$.

5. Show that $(0,0)$ is an asymptotically stable equilibrium for the system

$$\dot{x} = -2xy, \qquad \dot{y} = x^2 - y^3 + y^5,$$

and determine a domain of stability.

(Hint: Look for a suitable quadratic Liapunov function.)

6. Solve the initital value problem

$$(t^2 + 1)\,\ddot{x}(t) - 2x(t) = 0, \qquad x(0) = 0, \qquad \dot{x}(0) = 1.$$

(Hint: $x(t) = t^2 + 1$ satisfies the ODE.)

Solutions for TATA71 2019-04-24

1. The phase portrait is

$$\longleftarrow 0 \longrightarrow K \longleftarrow$$

and the general solution formula is

$$x(t) = \frac{K x_0}{x_0 + (K - x_0)\, e^{-rt}} = \frac{K x_0\, e^{rt}}{K + (e^{rt} - 1)\, x_0},$$

as can be found by separation of variables:

$$x = 0 \quad \text{or} \quad x = K \quad \text{or} \quad \int \frac{dx}{x(1 - x/K)} = \int r\, dt, \quad \text{etc.}$$

(Alternatively, set $x(t) = 1/y(t)$ to get a linear ODE for y.)

2. The phase portrait is a stable spiral (the trace of the system matrix is -1, the determinant is $5/2$). For graphics, type "streamplot {-5y/4, 2x-y}" in Wolfram Alpha (or click on the link).

The fastest way of deriving the solution formula is perhaps to write the system as a single second-order ODE $\ddot{x} + \dot{x} + \frac{5}{2}x = 0$, whose characteristic polynomial $\lambda^2 + \lambda + \frac{5}{2} = (\lambda + \frac{1}{2})^2 + \frac{9}{4}$ gives the general solution

$$x(t) = e^{-t/2}\big(A\cos(\tfrac{3}{2}t) + B\sin(\tfrac{3}{2}t)\big),$$

so that

$$\begin{aligned} y(t) = -\tfrac{4}{5}\dot{x}(t) &= -\tfrac{4}{5}e^{-t/2}\Big(\big(-\tfrac{1}{2}A + \tfrac{3}{2}B\big)\cos(\tfrac{3}{2}t) + \big(-\tfrac{1}{2}B - \tfrac{3}{2}A\big)\sin(\tfrac{3}{2}t)\Big) \\ &= \tfrac{2}{5}e^{-t/2}\Big(\big(A - 3B\big)\cos(\tfrac{3}{2}t) + \big(3A + B\big)\sin(\tfrac{3}{2}t)\Big). \end{aligned}$$

The initial conditions $x(0) = 1$ and $y(0) = 0$ give $A = 1$ and $B = \frac{1}{3}$, so the particular solution that was asked for is

$$\begin{aligned} x(t) &= e^{-t/2}\big(\cos(\tfrac{3}{2}t) + \tfrac{1}{3}\sin(\tfrac{3}{2}t)\big), \\ y(t) &= \tfrac{4}{3}e^{-t/2}\sin(\tfrac{3}{2}t). \end{aligned}$$

For $0 \le t \le 2\pi$, the spiral goes one and a half lap ($\frac{3}{2} \cdot 2\pi = 3\pi$) around the origin. Graphics: "parametric plot (exp(-t/2) (cos(3t/2)+sin(3t/2)/3), exp(-t/2) sin(3t/2)*4/3), t=0..2*pi".

3. The x-nullcline is the parabola $y = x^2$, and the y-nullcline is the union of the lines $x = 1$ and $x = -2$. They intersect at the equilibrium points $(x, y) = (1, 1)$ and $(x, y) = (-2, 4)$. Jacobian matrix:

$$J(x, y) = \begin{pmatrix} -2x & 1 \\ 2x+1 & 0 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}, \qquad J(-2,4) = \begin{pmatrix} 4 & 1 \\ -3 & 0 \end{pmatrix}.$$

   For $(1,1)$, $\det(J) = -3 < 0$, so it's a **saddle point** (with eigenvalues -3 and 1, principal directions $(-1,1)^T$ and $(1,3)^T$).

   For $(-2,4)$, $\beta = \mathrm{tr}(J) = 4$ and $\gamma = \det(J) = 3$, which lies below the parabola $\gamma = (\beta/2)^2$, so it's an **unstable node** (with eigenvalues 3 and 1, principal directions $(-1,1)^T$ and $(-1,3)^T$).

   Phase portrait: "streamplot {y-x^2, x^2+x-2}, x=-5..5, y=-5..5".

4. Let $\beta = \mathrm{tr}(A)$ and $\gamma = \det(A)$. Since $\gamma \neq 0$ by assumption, $(x, y) = (0,0)$ is the only equilibrium point, and we know that it is asymptotically stable iff the eigenvalues of A have negative real part, and neutrally stable iff they lie on the imaginary axis. The eigenvalues are the roots of the characteristic polynomial $\det(A - \lambda I) = \lambda^2 - \beta\lambda + \gamma$:

$$\lambda_{1,2} = \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 - \gamma}.$$

   If $\gamma < 0$, then the square root is real, and greater than $|\beta/2|$, so in this case there is one negative and one positive eigenvalue, and the origin is unstable (a saddle point). If $\gamma > 0$, either the square root is imaginary, or it is real but smaller than $|\beta/2|$, so in this case the real parts of λ_1 and λ_2 both have the same sign as $\beta/2$; thus, the origin is asymptotically stable if $\beta < 0$, neutrally stable if $\beta = 0$ and unstable if $\beta > 0$.

   In conclusion, the criterion is that the origin is asymptotically stable if $\mathrm{tr}(A) < 0$ and $\det(A) > 0$, neutrally stable if $\mathrm{tr}(A) = 0$ and $\det(A) > 0$, and unstable otherwise.

5. Try $V(x, y) = ax^2 + by^2$. The choice $V(x, y) = x^2 + 2y^2$ works; it's positive definite, and

$$\dot{V} = V_x\dot{x} + V_y\dot{y} = 2x \cdot (-2xy) + 4y \cdot (x^2 - y^3 + y^5) = -4y^4(1 - y^2)$$

   is negative semidefinite in the strip $-1 < y < 1$, so V is a weak Liapunov function in that strip. We have $V = 0$ along the x-axis, but the vector field is $(\dot{x}, \dot{y}) = (0, 2x^2)$ when $y = 0$, so it points out from the x-axis. Thus, the only trajectory which is contained in the x-axis is the equilibrium point $(0,0)$ itself, and therefore LaSalle's theorem shows that this equilibrium is asymptotically stable.

For a domain of stability, take the largest sub-level set of V that's contained in the strip: the elliptical region

$$\{(x,y) \in \mathbf{R}^2 : x^2 + 2y^2 < 2\}.$$

(Phase portrait: "streamplot {-2 x y, x^2-y^3+y^5}, x=-2..2, y=-2..2".)

6. We use reduction of order to find a second solution, linearly independent of the given solution $x_0(t) = t^2 + 1$. With $x(t) = x_0(t)\,Y(t)$ and $y(t) = \dot{Y}(t)$, the ODE becomes

$$\begin{aligned} 0 &= (t^2+1)(\ddot{x}_0\,Y + 2\dot{x}_0\,\dot{Y} + x_0\,\ddot{Y}) - 2x_0\,Y \\ &= \underbrace{\big((t^2+1)\ddot{x}_0 - 2x_0\big)}_{=0} Y + 2(t^2+1)\dot{x}_0\,\dot{Y} + (t^2+1)x_0\,\ddot{Y} \\ &= (t^2+1)\Big(2\cdot 2t\,y + (t^2+1)\,\dot{y}\Big), \end{aligned}$$

that is,

$$\dot{y} + \frac{4t}{t^2+1}\,y = 0.$$

Multiplication by the integrating factor $\exp\big(\int \frac{4t}{t^2+1}\,dt\big) = \exp(2\ln(t^2+1)) = (t^2+1)^2$ gives

$$\tfrac{d}{dt}\Big((t^2+1)^2\,y(t)\Big) = 0,$$

so $y(t) = C/(t^2+1)^2$ for some constant C, let's say $C = 1$ since we're only looking for one solution. Then

$$Y(t) = \int y(t)\,dt = \int \frac{dt}{(t^2+1)^2} = \frac{1}{2}\left(\frac{t}{t^2+1} + \arctan t\right) \quad (+\text{constant}).$$

(For the computation of this antiderivative, see any calculus textbook, for instance Forsling & Neymark, *Matematisk analys: En variabel*, Ex. 5.30.)

Thus, the second solution is $x(t) = x_0(t)\,Y(t) = \frac{1}{2}\big(t + (t^2+1)\arctan t\big)$, which happens to satisfy the given initial conditions already, as is seen by a glance at the Maclaurin expansion $x(t) = 0 + t + O(t^2)$. (In general, one would have to pick a suitable linear combination of the two solutions.)

Answer. $x(t) = \frac{1}{2}\big(t + (t^2+1)\arctan t\big)$.