Linköpings universitet
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Kurskod: TATA71
Provkod: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2019-08-27 kl. 8.00–13.00

No aids allowed. You may write your answers in English or Swedish (or both). Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points. Solutions will be posted on the course webpage afterwards. Good luck!

1. (a) Draw the phase portrait for the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \qquad r > 0, \qquad K > 0.$$

   (b) Consider a model for "logistic population growth with harvesting":

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - E, \qquad E \geq 0.$$

   How does the phase portrait change as the value of the parameter E increases from 0? At what value of E does the phase portrait start to look qualitatively different from the original case $E = 0$?

2. Sketch the phase portrait for the system

$$\dot{x} = (x-2)(y-x^2), \qquad \dot{y} = y - 1,$$

   and use linearization to classify the equilibrium points.

3. Determine k such that $x(t) = e^{kt}$ satisfies the ODE

$$(t^2+1)\,\ddot{x}(t) - (t^2+2t+1)\,\dot{x}(t) + 2t\,x(t) = 0,$$

   and then use reduction of order to find the general solution.

4. Rewrite the system

$$\dot{x} = x(1-x^2-y^2)(x^2+y^2-4) - y(x^2+y^2),$$
$$\dot{y} = y(1-x^2-y^2)(x^2+y^2-4) + x(x^2+y^2)$$

   in terms of polar coordinates, and use this to determine whether there are any stable or unstable limit cycles.

5. Compute the general (real-valued) solution of the linear system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 1 \\ 8 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

6. Show that the origin is a globally asymptotically stable equilibrium for the system

$$\dot{x} = xy^4 - 2x^3 - y, \qquad \dot{y} = 2x + 2x^2y^3 - y^7.$$

Solutions for TATA71 2019-08-27

1. (a) Phase portrait:

$$\longleftarrow 0 \longrightarrow K \longleftarrow$$

   (b) The right-hand side in the logistic equation, $f(x) = rx(1 - x/K)$, is a quadratic polynomial with zeros $x = 0$ and $x = K$, and maximum $f(K/2) = rK/4$. Subtracting E shifts the graph of f downwards. For $0 < E < rK/4$, there will still be two real zeros $x_{1,2}$ (but closer together), so the phase portrait qualitatively looks the same:

$$\longleftarrow x_1 \longrightarrow x_2 \longleftarrow$$

   But at $E = rK/4$ there is just a single zero x_0:

$$\longleftarrow x_0 \longleftarrow$$

   And for $E > rK/4$ there are no real zeros:

$$\longleftarrow$$

   Answer. The phase portrait changes its character at $E = rK/4$.

2. The equilibria are given by $(x-2)(y-x^2) = 0$ and $y-1 = 0$, so $y = 1$ to begin with, and then $x = 2$ or $x = \pm 1$, so $(x, y) = (2, 1)$ or $(x, y) = (\pm 1, 1)$. Evaluating the Jacobian $J(x, y) = \begin{pmatrix} y-x^2-2x(x-2) & x-2 \\ 0 & 1 \end{pmatrix}$ at the equilibria gives $J(2, 1) = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$ (saddle point), $J(1, 1) = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ (unstable node) and $J(-1, 1) = \begin{pmatrix} -6 & -3 \\ 0 & 1 \end{pmatrix}$ (saddle point).

   Phase portrait: "streamplot {(x-2)(y-x^2),y-1}, x=-3..3, y=-3..3" in Wolfram Alpha. Note that the lines $x = 2$ and $y = 1$ are invariant.

3. Plugging $x = e^{kt}$ into the ODE, one finds quickly that $k = 1$ is the only value that works. So $x_0(t) = e^t$ is a solution. Let $x(t) = Y(t)\, x_0(t) = Y(t)\, e^t$ and $y(t) = \dot{Y}(t)$. This gives

$$\begin{aligned} 0 &= (t^2+1)\,\ddot{x} - (t^2+2t+1)\,\dot{x} + 2t\,x \\ &= (t^2+1)\,(\ddot{Y} + 2\dot{Y} + Y)e^t - (t^2+2t+1)\,(\dot{Y} + Y)e^t + 2t\,Ye^t \\ &= e^t\Big((t^2+1)\,\ddot{Y} + (t^2-2t+1)\,\dot{Y}\Big) = e^t\Big((t^2+1)\,\dot{y} + (t^2-2t+1)\,y\Big), \end{aligned}$$

   that is,

$$\dot{y} + \left(1 - \frac{2t}{t^2+1}\right) y = 0.$$

Multiplication by the integrating factor $\exp(t - \ln(t^2+1)) = e^t/(t^2+1)$ gives

$$\frac{d}{dt}\left(\frac{e^t}{t^2+1}y(t)\right) = 0 \qquad \Longleftrightarrow \qquad y(t) = A(t^2+1)e^{-t},$$

and thus

$$Y(t) = \int y(t)\,dt = A\int (t^2+1)e^{-t}dt = -A(t^2+2t+3)e^{-t} + B,$$

so that the general solution is (if we let $C = -A$ for cosmetic reasons)

$$x(t) = Y(t)e^t = Be^t + C(t^2+2t+3).$$

4. As a first step, write

$$\dot{x} = x(1-r^2)(r^2-4) - yr^2,$$
$$\dot{y} = y(1-r^2)(r^2-4) + xr^2.$$

This gives

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = r(1-r^2)(r^2-4)$$

and

$$\dot{\theta} = \frac{\dot{y}x - y\dot{x}}{r^2} = r^2.$$

Since $\dot{\theta} > 0$ for $r > 0$, the motion goes counterclockwise around the origin (which obviously is an equilibrium). The one-dimensional phase portrait for the r-equation is (for $r \geq 0$) "$0 \leftarrow 1 \rightarrow 2 \leftarrow$", which shows that the circle $x^2 + y^2 = 1$ is an unstable limit cycle and the circle $x^2 + y^2 = 4$ is a stable limit cycle.

5. The system matrix

$$A = \begin{pmatrix} 5 & -1 & 1 \\ 8 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

has characteristic polynomial

$$\det(A - \lambda I) = \lambda^3 - 7\lambda^2 + 19\lambda - 13 = (\lambda - 1)(\lambda^2 - 6\lambda + 13),$$

so the eigenvalues are $3 \pm 2i$ and 1, with eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

respectively. So the change of variables

$$\mathbf{x} = M\mathbf{y}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

brings the system $\dot{\mathbf{x}} = A\mathbf{x}$ to Jordan normal form

$$\dot{\mathbf{y}} = M^{-1}AM\mathbf{y} = J\mathbf{y}, \qquad J = \begin{pmatrix} 3 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the general solution

$$\mathbf{y}(t) = \begin{pmatrix} e^{3t}(A\cos 2t + B\sin 2t) \\ e^{3t}(A\sin 2t - B\sin 2t) \\ Ce^{t} \end{pmatrix},$$

so the answer is

$$\begin{aligned} \mathbf{x}(t) = M\mathbf{y}(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t}(A\cos 2t + B\sin 2t) \\ e^{3t}(A\sin 2t - B\cos 2t) \\ Ce^{t} \end{pmatrix} \\ &= Ae^{3t}\begin{pmatrix} \cos 2t \\ 2\cos 2t + \sin 2t \\ -\sin 2t \end{pmatrix} + Be^{3t}\begin{pmatrix} \sin 2t \\ 2\sin 2t - \cos 2t \\ \cos 2t \end{pmatrix} + Ce^{t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \end{aligned}$$

where A, B and C are arbitrary real constants.

6. $V(x,y) = 2x^2 + y^2$ is a weak Liapunov function, since it is positive definite and

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 4x(xy^4 - 2x^3 - y) + 2y(2x + 2x^2y^3 - y^7) \\ &= 8x^2y^4 - 8x^4 - 2y^8 = -2(2x^2 - y^4)^2 \le 0. \end{aligned}$$

The set where $\dot{V} = 0$ consists of the two curves $x = \pm y^2/\sqrt{2}$, which can be parametrized as $(x,y) = (\pm s^2/\sqrt{2}, s)$. The tangent vector at a typical point on one of these curves is $(\frac{dx}{ds}, \frac{dy}{ds}) = (\pm\sqrt{2}s, 1)$, and the vector field $(\dot{x},\dot{y}) = \big(x(y^4 - 2x^2) - y, 2x + (2x^2 - y^4)y^3\big)$ reduces to $(\dot{x},\dot{y}) = (-y, 2x) = (-s, \pm\sqrt{2}s^2)$, which is orthogonal to the tangent vector (and nonzero for $s \neq 0$). This shows that the trajectories, except for the equilibrium point $(0,0)$ itself, do not stay on the curves where $\dot{V} = 0$, and asymptotic stability therefore follows from LaSalle's theorem. For *global* asymptotic stability, it is enough to remark that (in addition to the above) $V(x,y) \to \infty$ as $|(x,y)| \to \infty$.