Linköpings universitet
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2020-03-17 kl. 14.00–19.00

No aids allowed. You may write your answers in English or Swedish (or both).

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Sketch the phase portrait for the ODE $\dot{x} = x^3$, and compute the flow $\phi_t(x)$.

2. Show that the origin is a globally stable equilibrium point of the system

$$\dot{x} = -x + 6y^3 - 3y^4, \qquad \dot{y} = -x - y + \tfrac{1}{2}xy.$$

   (Hint: Look for a Liapunov function of the form $x^2 + cy^k$.)

3. Compute the general solution of the linear system

$$\dot{x} = x, \qquad \dot{y} = y - 2x,$$

   and in particular compute the solution satisfying $\big(x(0), y(0)\big) = (1,0)$. Also sketch the phase portrait.

4. State and prove the trace–determinant criterion for stability of a simple 2×2 linear system $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^2$, $\det(A) \neq 0$.

5. Find a constant of motion for the system

$$\dot{x} = y - x^2, \qquad \dot{y} = 2x(y-1)$$

   and sketch the phase portrait.

6. Use variation of constants to compute the general solution of $\ddot{x} + x = \tan t$.

Solutions for TATA71 2020-03-17

1. Phase portrait for $\dot{x} = x^3$: $\longleftarrow 0 \longrightarrow$

   To solve the ODE, consider the equilibrium solution $x(t) = 0$ separately. All other solutions are given by $\dot{x}\,x^{-3} = 1$, which integrates to $-\frac{1}{2}x^{-2} = t + C$. With $x(0) = x_0 \neq 0$, we thus get $-\frac{1}{2}x(t)^{-2} = t - \frac{1}{2}x_0^{-2}$, so that $x(t)^2 = x_0^2/(1 - 2tx_0^2)$. Choosing the correct sign when taking square roots (so that we get $x(0) = x_0$ and not $x(0) = -x_0$) gives $x(t) = x_0/(1-2tx_0^2)^{1/2}$, and this formula gives the correct solution also in the exceptional case $x_0 = 0$.

   Answer. The flow is $\phi_t(x) = \frac{x}{\sqrt{1-2tx^2}}$ (for all $t \in \mathbf{R}$ if $x = 0$, for $t < (2x^2)^{-1}$ if $x \neq 0$).

2. With $V(x, y) = x^2 + cy^k$ we have

$$\begin{aligned}
\dot{V} &= V'_x\dot{x} + V'_y\dot{y} \\
&= 2x(-x + 6y^3 - 3y^4) + cky^{k-1}(-x - y + \tfrac{1}{2}xy) \\
&= -2x^2 + 12xy^3 - 6xy^4 - ckxy^{k-1} - cky^k + \tfrac{1}{2}ckxy^k \\
&= (-2x^2 - cky^k) + x(12y^3 - cky^{k-1}) - \tfrac{1}{2}xy(12y^3 - cky^{k-1}).
\end{aligned}$$

   Taking $k = 4$ and $c = 3$, we get a positive definite function $V = x^2 + 3y^4$ such that $\dot{V} = -2x^2 - 12y^4$ is negative definite. So V is a strong Liapunov function for the system. Moreover, $V(x, y) \to \infty$ as $\sqrt{x^2 + y^2} \to \infty$, so the origin is a *globally* stable equilibrium.

3. We can integrate the first equation $\dot{x} = x$ immediately: $x(t) = Ae^t$. Then the second equation $\dot{y} = y - 2x$ becomes $\dot{y} - y = -2Ae^t$, or $\frac{d}{dt}(ye^{-t}) = -2A$, so that $y(t) = (-2At + B)e^t$. The initial conditions $\big(x(0), y(0)\big) = (1, 0)$ correspond to $A = 1$ and $B = 0$.

   Phase portrait: "streamplot {x,y-x}, x=-3..3, y=-3..3" in Wolfram Alpha. The origin is an unstable improper node.

   Answer. General solution $\big(x(t), y(t)\big) = (Ae^t, (B - 2At)e^t)$. Particular solution $\big(x(t), y(t)\big) = (e^t, -2te^t)$.

4. Let $\beta = \operatorname{tr}(A)$ and $\gamma = \det(A)$. Since $\gamma \neq 0$ by assumption, $(x, y) = (0, 0)$ is the only equilibrium point, and we know that it is asymptotically stable iff the eigenvalues of A have negative real part, and neutrally stable iff they lie on the imaginary axis. The eigenvalues are the roots of the characteristic polynomial $\det(A - \lambda I) = \lambda^2 - \beta\lambda + \gamma$:

$$\lambda_{1,2} = \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 - \gamma}.$$

If $\gamma < 0$, then the square root is real, and greater than $|\beta/2|$, so in this case there is one negative and one positive eigenvalue, and the origin is unstable (a saddle point). If $\gamma > 0$, either the square root is imaginary, or it is real but smaller than $|\beta/2|$, so in this case the real parts of λ_1 and λ_2 both have the same sign as $\beta/2$; thus, the origin is asymptotically stable if $\beta < 0$, neutrally stable if $\beta = 0$ and unstable if $\beta > 0$.

In conclusion, the criterion is that the origin is asymptotically stable if $\mathrm{tr}(A) < 0$ and $\det(A) > 0$, neutrally stable if $\mathrm{tr}(A) = 0$ and $\det(A) > 0$, and unstable otherwise.

5. The system has the Hamiltonian form $\dot{x} = \partial H/\partial y$, $\dot{y} = -\partial H/\partial x$ where $H(x, y) = x^2 + \frac{1}{2}y^2 - x^2 y$, and H is therefore automatically a constant of motion.

   Phase portrait: "streamplot {y-x^2,2x(y-1)}, x=-3..3, y=-3..3" in Wolfram Alpha.

   The origin is a neutrally stable equilibrium, since it is surrounded by closed orbits that follow the level sets of H (near the origin these level sets resemble ellipses $x^2 + \frac{1}{2}y^2 = C$). The equilibria $(\pm 1, 1)$ are saddle points, by the trace–determinant criterion.

6. Write the equation $\ddot{x} + x = \tan t$ as a system, by letting $x_1 = x$ and $x_2 = \dot{x}$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tan t \end{pmatrix}.$$

   From $x_{\mathrm{hom}}(t) = A\cos t + B\sin t$ we compute the fundamental matrix

$$\Phi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

   and we make the substitution $\mathbf{x(t)} = \Phi(t)\mathbf{y(t)}$ as usual. Then the system becomes

$$\dot{\mathbf{y}} = \Phi^{-1} \begin{pmatrix} 0 \\ \tan t \end{pmatrix} = \begin{pmatrix} -\sin^2 t/\cos t \\ \sin t \end{pmatrix}.$$

   Integration gives

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sin t + \frac{1}{2}\ln\left|\frac{1-\sin t}{1+\sin t}\right| + A \\ -\cos t + B \end{pmatrix},$$

   from which we can compute $x(t) = x_1(t)$ from the first row in the matrix product $\mathbf{x(t)} = \Phi(t)\mathbf{y(t)}$:

$$x(t) = \cos t \cdot y_1(t) + \sin t \cdot y_2(t).$$

   Answer. $x(t) = A\cos t + B\sin t + \frac{1}{2}\cos t \cdot \ln\left|\frac{1-\sin t}{1+\sin t}\right|$.