Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

# TATA71 Ordinära differentialekvationer och dynamiska system

## Tentamen 2022-08-26 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least $n$ passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. The following equation has been used as a model for insect populations (where the last term represents predation by birds):
$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - \frac{bN^2}{a^2+N^2} \qquad (r,K,a,b>0)$$
Show how to rescale the variables to obtain the dimensionless ODE
$$\frac{dn}{d\tau} = \alpha n\left(1-\frac{n}{\beta}\right) - \frac{n^2}{1+n^2}.$$
State clearly how the new variables $(n,\tau)$ and parameters $(\alpha,\beta)$ are defined in terms of the original variables $(N,t)$ and parameters $(r,K,a,b)$.

2. Compute the general solution of the linear system
$$\dot{x} = -x+3y, \qquad \dot{y} = -y,$$
and draw the phase portrait.

3. Investigate stability of equilibria, and sketch the phase portrait, for the system
$$\dot{x} = x+x^3-y, \qquad \dot{y} = x(y-2).$$

4. Show that $V(x,y) = x^2+y^2$ is a strong Liapunov function for the system
$$\dot{x} = y-x^3, \qquad \dot{y} = -x+y^3(x^2-1)$$
in the region $|x|<1$. Use this to show that the origin is an asymptotically stable equilibrium and to determine a domain of stability.

5. Let $\mathbf{x}^*$ be an equilibrium point for the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$. Give the precise definitions of the following concepts:
   (a) $\mathbf{x}^*$ is **stable**.
   (b) $\mathbf{x}^*$ is **asymptotically stable**.
   (c) $\mathbf{x}^*$ is **globally asymptotically stable**.

6. Determine the general solution of the ODE
$$\ddot{x}(t) + 4x(t) = \frac{1}{\cos 2t}, \qquad |t| < \frac{\pi}{4}.$$

# Solutions for TATA71 2022-08-26

1. With $N = c_1 n$ and $t = c_2 \tau$ the equation becomes

$$\frac{dn}{d\tau} = \frac{c_2}{c_1}\left(r c_1 n\left(1 - \frac{c_1 n}{K}\right) - \frac{b c_1^2 n^2}{a^2 + c_1^2 n^2}\right).$$

   Now let $c_1 = a$ och $c_2 = a/b$ to obtain

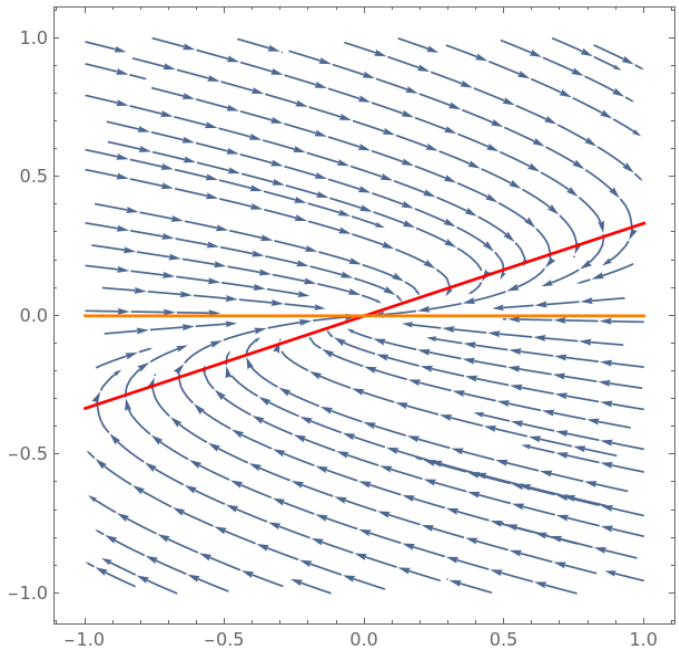
$$\frac{dn}{d\tau} = \alpha n\left(1 - \frac{n}{\beta}\right) - \frac{n^2}{1+n^2},$$

   where $\alpha = c_2 r = ar/b$ and $\beta = K/c_1 = K/a$.

   **Answer.** Dimensionless variables: $n = N/a$ and $\tau = bt/a$. Dimensionless parameters: $\alpha = ar/b$ and $\beta = K/a$.

2. The second equation $\dot{y} = -y$ immediately gives $y(t) = Ce^{-t}$, and then the first equation $\dot{x} = -x + 3y$ reads $\dot{x} + x = 3Ce^{-t}$, which is equivalent to $\frac{d}{dt}\left(x(t)e^{t}\right) = 3C$, so that $x(t) = (3Ct + D)e^{-t}$. The origin is a stable improper node with principal direction $(1,0)^T$ and nullclines $x = 3y$ and $y = 0$.

   **Answer.** The general solution is $x(t) = (3Ct + D)e^{-t}$, $y(t) = Ce^{-t}$, where $C$ and $D$ are arbitrary real constants. The phase portrait is shown below.



3. We have $\dot{y} = x(y-2) = 0$ iff $x = 0$ or $y = 2$. Inserting this into the equation $\dot{x} = x + x^3 - y = 0$ we get at once $y = 0$ when $x = 0$, and when $y = 2$ we have $x + x^3 = 2$; this equation has an obvious root $x = 1$, and it is the only one, since $x + x^3$ is an increasing function of $x$. Hence the equilibrium points are $(0,0)$ and $(1,2)$.

   The Jacobian matrix of the system is

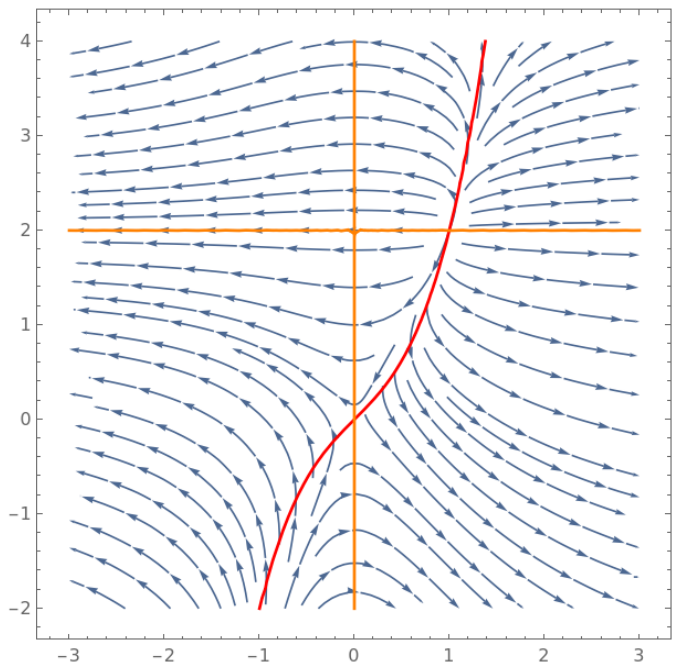
$$J(x,y) = \begin{pmatrix} 1+3x^2 & -1 \\ y-2 & x \end{pmatrix},$$

   so at the equilibria we have

$$J(0,0) = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}, \qquad J(1,2) = \begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix}.$$

   Thus $(0,0)$ is a **saddle** (eigenvalues 2 and $-1$ with eigenvectors $(1,-1)^T$ and $(1,2)^T$), while $(1,2)$ is an **unstable node** (eigenvalues 4 and 1 with eigenvectors $(1,0)^T$ and $(1,3)^T$).

   **Answer.** Saddle at $(0,0)$, unstable node at $(1,2)$. The phase portrait is shown below.



4. We compute

$$\begin{aligned}\dot{V} &= V_x\dot{x} + V_y\dot{y}\\ &= 2x\left(y - x^3\right) + 2y\left(-x + y^3(x^2-1)\right)\\ &= -2x^4 - 2y^4(1-x^2),\end{aligned}$$

so $\dot{V}$ is negative definite in the strip $\Omega = \{|x| < 1\}$ (i.e., if $|x| < 1$ and $(x,y) \neq (0,0)$, then $\dot{V} < 0$). Moreover $V$ is obviously positive definite (in all of $\mathbf{R}^2$), so it is indeed a strong Liapunov function in $\Omega$. By the strong Liapunov theorem, this immediately implies that the origin is asymptotically stable. Any sublevelset of $V$ contained in $\Omega$, i.e., any disk $x^2+y^2 \leq r^2$ with $0 < r < 1$, provides a domain of stability, and we can take the union of all these disks to show that the unit disk $x^2 + y^2 < 1$ is a domain of stability.

5. (a) $\mathbf{x}^*$ is stable if for every neighbourhood $U$ of $\mathbf{x}^*$ there is neighbourhood $V$ of $\mathbf{x}^*$ such that every trajectory starting in $V$ stays in $U$ for all $t \geq 0$.

   (b) $\mathbf{x}^*$ is asymptotically stable if it is stable and has a neighbourhood $W$ such that every trajectory starting in $W$ converges to $\mathbf{x}^*$ as $t \to \infty$.

   (c) $\mathbf{x}^*$ is globally asymptotically stable if it is stable and every trajectory (starting anywhere in $\mathbf{R}^n$) converges to $\mathbf{x}^*$ as $t \to \infty$.

6. The general solution of the homogeneous equation $\ddot{x}(t)+4x(t) = 0$ is $x(t) = A\cos 2t + B\sin 2t$, so the corresponding first-order system obtained by letting $x_1 = x$ and $x_2 = \dot{x}$ has the general solution $x_1(t) = A\cos 2t + B\sin 2t$, $x_2(t) = \dot{x}_1(t) = -2A\sin 2t + 2B\cos 2t$. This gives the fundamental matrix

$$\Phi(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix}.$$

For the inhomogeneous system we let $\mathbf{x} = \Phi\mathbf{y}$ to obtain

$$\begin{aligned}\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ 1/\cos 2t \end{pmatrix} \quad &\Longleftrightarrow \quad \Phi\dot{\mathbf{y}} = \begin{pmatrix} 0 \\ 1/\cos 2t \end{pmatrix}\\ \Longleftrightarrow \quad \dot{\mathbf{y}} = \frac{1}{2}\begin{pmatrix} 2\cos 2t & -\sin 2t \\ 2\sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} 0 \\ 1/\cos 2t \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} -\sin 2t/\cos 2t \\ 1 \end{pmatrix}.\end{aligned}$$

Integration (using $\cos 2t > 0$ for $|t| < \pi/4$) now gives

$$\mathbf{y} = \begin{pmatrix} \frac{1}{4}\ln(\cos 2t) + A \\ \frac{1}{2}t + B \end{pmatrix}$$

and then $x(t) = x_1(t) = y_1(t)\cos 2t + y_2(t)\sin 2t$.

**Answer.** $x(t) = A\cos 2t + B\sin 2t + \frac{1}{4}\cos 2t\ln(\cos 2t) + \frac{1}{2}t\sin 2t$, for $|t| < \frac{\pi}{4}$.