Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2025-01-16 kl. 14.00–19.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Consider the population model

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right)\left(\frac{x}{A}-1\right), \qquad \text{where } 0<r \text{ and } 0<A<K.$$

   (a) Draw the phase portrait (for $x \geq 0$).

   (b) Show how to rescale the variables x and t in order to obtain the dimensionless system $\frac{dy}{d\tau} = y(1-y)\left(\frac{y}{\alpha}-1\right)$. How is the new parameter α defined in terms of the original parameters (r,K,A)?

2. Determine the general solution of the linear system

$$\dot{x} = 3y, \qquad \dot{y} = x - 2y,$$

   and draw the phase portrait as carefully as you can.

3. Use linearization to classify the equilibrium points of the system

$$\dot{x} = x - y, \qquad \dot{y} = (x-1)\,y,$$

   and sketch the phase portrait.

4. Sketch the phase portrait of the system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = 5\begin{pmatrix}y\\ -x\end{pmatrix} + (4-x^2-y^2)\begin{pmatrix}x\\ y\end{pmatrix}$$

   by expressing it in polar coordinates.

5. Show that the origin is an asymptotically stable equilibrium for the system

$$\dot{x} = -y + xy^4, \qquad \dot{y} = x - y^3,$$

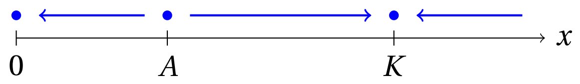
   and determine a domain of stability. (Hint: Try a very commonly used Liapunov function!)

6. Solve the linear ODE

$$\ddot{x}(t) - 2\dot{x}(t) + x(t) = \frac{e^{2t}}{(1+e^t)^2}.$$

Solutions for TATA71 2025-01-16

1. (a) Phase portrait for $x \geq 0$:



Remark. This model features an *underpopulation effect*, where the per-capita growth rate $\dot{x}/x$ becomes smaller (in this case even negative) at low population levels. For instance, it may be difficult to find mates.

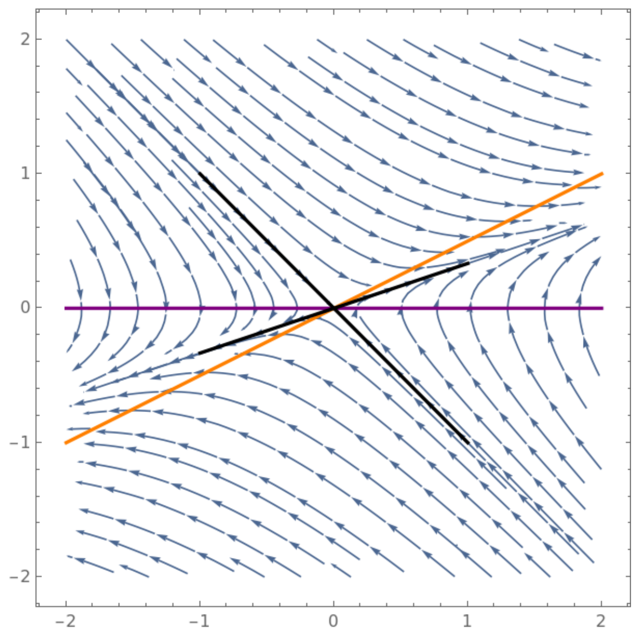
   (b) With $\tau = rt$ and $y = x/K$ the ODE becomes

$$\frac{dy}{d\tau} = y(1-y)\left(\frac{y}{A/K} - 1\right),$$

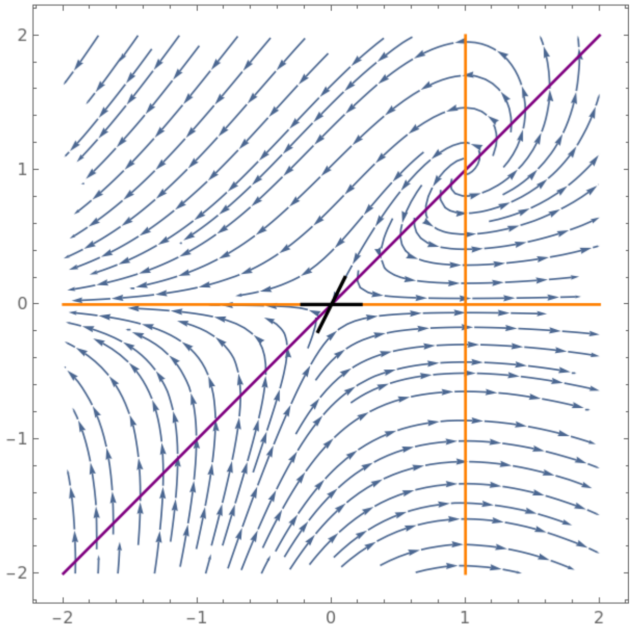
which has the desired form with $\alpha = A/K$ (so that $0 < \alpha < 1$).
(The alternative choice $\tau = rt$, $y = x/A$ and $\alpha = K/A > 1$ also works.)

2. The system's matrix $A = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and from this information we can write down the solution immediately. Since $\lambda_2 < 0 < \lambda_1$, the equilibrium at the origin is a saddle, with principal directions given by the eigenvectors.

Answer. The general solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{t} + B \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$, where A and B are arbitrary constants. Phase portrait (nullclines $\dot{x} = 3y = 0$ in purple and $\dot{y} = x - 2y = 0$ in orange, principal directions in black):



3. The equilibrium points are $(x,y) = (0,0)$ and $(x,y) = (1,1)$. The Jacobian matrix is $J(x,y) = \begin{pmatrix} 1 & -1 \\ y & x-1 \end{pmatrix}$. Since $J(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ has eigenvalues ± 1 of opposite signs, there is a **saddle** at the origin, with principal directions given by the eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. At $(1,1)$ there is an **unstable focus** according to the trace–determinant criterion, since $J(1,1) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ gives $\beta = \operatorname{tr} J = 1 > 0$ and $\gamma = \det J = 1 > (\beta/2)^2$. Phase portrait:

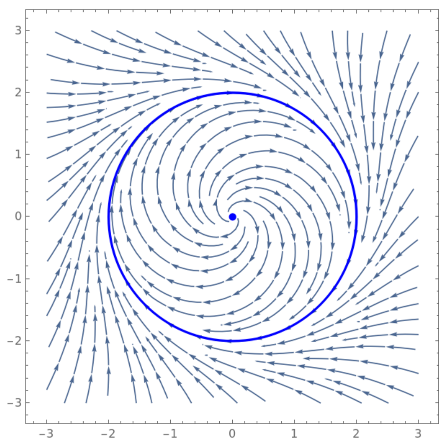


4. For $r > 0$ we have, according to the usual formulas,

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{(4 - x^2 - y^2)(x^2 + y^2)}{r} = (4 - r^2)r,$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{-5(x^2 + y^2)}{r^2} = -5,$$

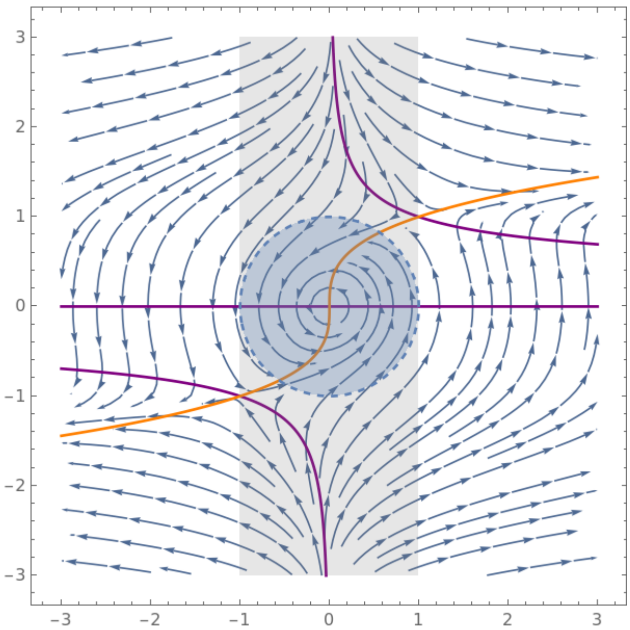
so the solution curves go clockwise around the equilibrium point at the origin, with r increasing for $0 < r < 2$ and decreasing for $2 < r$. Hence, the circle $x^2 + y^2 = 2^2$ is a stable limit cycle. Phase portrait:



5. With $V(x,y) = x^2 + y^2$ (which of course is positive definite) we have $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y + xy^4) + 2y(x - y^3) = 2(x^2 - 1)y^4$, which is negative semidefinite in the open strip $\Omega = (-1,1) \times \mathbf{R}$ (drawn in gray in the figure below). So V is a weak Liapunov function on Ω. The set of points in Ω where $\dot{V} = 0$ is the line segment $C = \{(x,0) : -1 < x < 1\}$. The only complete trajectory contained in C is the equilibrium solution at the origin, since the vector field reduces to $(\dot{x}, \dot{y}) = (0, x)$ when $y = 0$ and hence is transversal to C away from the origin. Thus LaSalle's theorem shows that the origin is asymptotically stable.

   The usual arguments show that the disk $x^2 + y^2 < k$ is a domain of stability for any $0 < k < 1$, and therefore so is the union of all these disks, namely the open unit disk $x^2 + y^2 < 1$ (drawn in blue below).

   As the following computer-draws phase portrait shows, the unit disk is not the largest possible domain of attraction, but it's what we get with this choice of Liapunov function:



6. The characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so the general solution of the homogeneous equation is $x_{\text{hom}}(t) = (At + B)e^t$.

We can use variation of constants to find a particular solution $x_{\text{part}}(t)$. With $(x_1, x_2) = (x, \dot{x})$ the ODE can be written as a system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2t}(1 + e^t)^{-2} \end{pmatrix}.$$

Since we found above that te^t and e^t form a basis of the solution space of the homogeneous equation, we can obtain a fundamental matrix for the system as follows:

$$\Phi(t) = \begin{pmatrix} te^t & e^t \\ \frac{d}{dt}(te^t) & \frac{d}{dt}(e^t) \end{pmatrix} = e^t \begin{pmatrix} t & 1 \\ t+1 & 1 \end{pmatrix}.$$

Letting $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$ leads in the usual way to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \Phi(t)^{-1} \begin{pmatrix} 0 \\ e^{2t}(1 + e^t)^{-2} \end{pmatrix} = e^{-t} \begin{pmatrix} -1 & 1 \\ t+1 & -t \end{pmatrix} \begin{pmatrix} 0 \\ e^{2t}(1 + e^t)^{-2} \end{pmatrix} = \frac{e^t}{(1+e^t)^2} \begin{pmatrix} 1 \\ -t \end{pmatrix},$$

which gives $y_1 = \int \frac{e^t\,dt}{(1+e^t)^2} = -\frac{1}{1+e^t} + A$ and (using integration by parts) $y_2 = \int t \cdot \frac{-e^t}{(1+e^t)^2}\,dt = t \cdot \frac{1}{1+e^t} - \int 1 \cdot \frac{1}{1+e^t}\,dt = \frac{t}{1+e^t} + \int \frac{-e^{-t}}{e^{-t}+1}\,dt = \frac{t}{1+e^t} + \ln(e^{-t}+1) + B$. Here we can take $A = B = 0$ and obtain $x_{\text{part}}(t) = e^t\big(t \cdot y_1(t) + 1 \cdot y_2(t)\big) = e^t \ln(e^{-t} + 1)$ from the first component in the matrix product $\Phi(t)\,\mathbf{y}(t)$.

Answer. The general solution is

$$x(t) = x_{\text{part}}(t) + x_{\text{hom}}(t) = e^t \ln(e^{-t} + 1) + (At + B)e^t,$$

where A and B are arbitrary constants.

Alternative method of solution. Actually, the quickest way of solving this ODE is probably to let $x(t) = z(t)e^t$ and use the exponential shift rule,

$$p(D)\big(z(t)e^{at}\big) = e^{at} p(D + a) z(t), \quad \text{where } D = \tfrac{d}{dt}.$$

Like this:

$$\begin{aligned}
& \ddot{x}(t) - 2\dot{x}(t) + x(t) = e^{2t}(1 + e^t)^{-2} \\
\iff\ & (D^2 - 2D + 1)x = e^{2t}(1 + e^t)^{-2} && \text{(express the LHS using the operator } D\text{)} \\
\iff\ & (D - 1)^2 x = e^{2t}(1 + e^t)^{-2} \\
\iff\ & (D - 1)^2 (ze^t) = e^{2t}(1 + e^t)^{-2} && \text{(let } x = ze^t\text{)} \\
\iff\ & e^t\big(D + 1 - 1\big)^2 z = e^{2t}(1 + e^t)^{-2} && \text{(use the shift rule, with } a = 1\text{)} \\
\iff\ & D^2 z = e^t(1 + e^t)^{-2} && \text{(cancel } e^t \text{ on both sides)} \\
\iff\ & Dz = -(1 + e^t)^{-1} + A = -e^{-t}(e^{-t} + 1)^{-1} + A && \text{(integrate)} \\
\iff\ & z = \ln(e^{-t} + 1) + At + B && \text{(integrate again)} \\
\iff\ & x = ze^t = \big(\ln(e^{-t} + 1) + At + B\big)e^t && \text{(go back to } x\text{)}.
\end{aligned}$$