Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2025-03-19 kl. 14.00–19.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Consider the logistic equation $\dot{x} = rx\bigl(1 - \frac{x}{K}\bigr)$, where $r > 0$ and $K > 0$.
   (a) Draw the phase portrait. (1p)
   (b) Solve the equation, with initial value $x(0) = x_0$. (2p)

2. Use linearization to classify the equilibrium points of the system
$$\dot{x} = x^2 + y^2 - 2, \qquad \dot{y} = x^2 - y,$$
and sketch the phase portrait.

3. Find all solutions to the ODE
$$\ddot{x}(t) - 4t\,\dot{x}(t) + (4t^2 - 2)\,x(t) = 0.$$
(Hint: $x(t) = e^{t^2}$ is a solution.)

4. Eliminate the time variable in the system
$$\dot{x} = y, \qquad \dot{y} = (y - x)\,y,$$
to get an ODE for y as a function of x, and use this to make an accurate drawing of the phase portrait.

5. Compute the matrix exponential e^{At}, where $A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$ and $t \in \mathbf{R}$.

6. Use the Lyapunov function $V(x,y) = x^2 + y^2$ to show that the origin is an asymptotically stable equilibrium for the system
$$\dot{x} = -y + xy - x^3 + x^4, \qquad \dot{y} = x - x^2 - y^3 + 3y^5,$$
and determine a domain of stability.

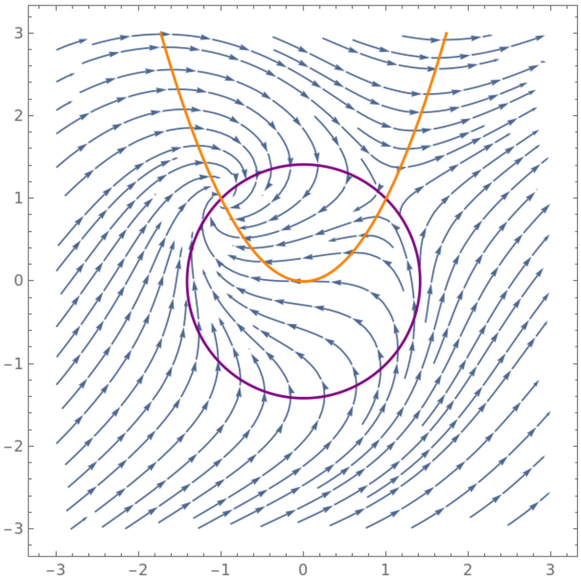
Solutions for TATA71 2025-03-19

1. For part (b), use separation of variables or let $y(t) = 1/x(t)$. (We have done this in class, so I omit the details here.)

   Answer. (a) $\longleftarrow 0 \longrightarrow K \longleftarrow$ (b) $x(t) = \dfrac{K x_0}{x_0 + (K - x_0)\, e^{-rt}}$.

2. The equilibria $(x, y) = (\pm 1, 1)$ are easily found. The Jacobian matrix is $J(x, y) = \begin{pmatrix} 2x & 2y \\ 2x & -1 \end{pmatrix}$, in particular $J(1, 1) = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$ and $J(-1, 1) = \begin{pmatrix} -2 & 2 \\ -2 & -1 \end{pmatrix}$. Since $\det J(1, 1)$ is negative, $(1, 1)$ is a saddle. (The eigenvalues are 3 and -2, with corresponding outgoing and incoming principal directions $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, respectively.) And from $\beta = \operatorname{tr} J(-1, 1) = -3 < 0$ and $\gamma = \det J(-1, 1) = 6 > (\beta/2)^2$, the trace–determinant criterion shows that $(-1, 1)$ is a stable focus.

   Phase portrait, with the nullclines in purple (for x) and orange (for y):



3. With $x = Ye^{t^2}$ we get $\dot{x} = \big(\dot{Y} + 2tY\big)e^{t^2}$ and $\ddot{x} = \big(\ddot{Y} + 4t\dot{Y} + (4t^2 + 2)Y\big)e^{t^2}$, which upon substitution into the ODE produces

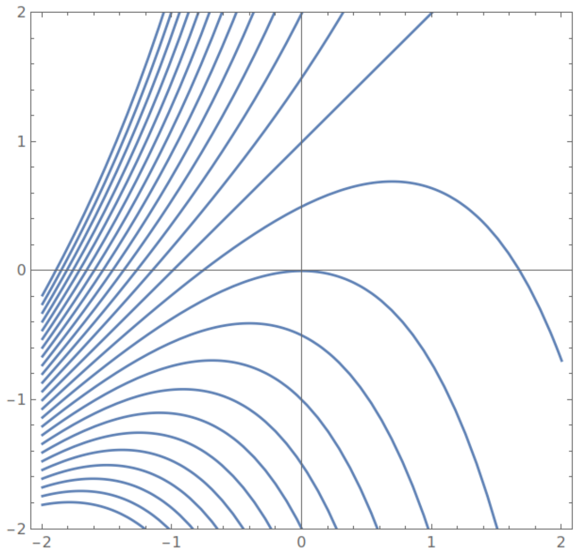
$$\begin{aligned} 0 &= \ddot{x} - 4t\,\dot{x} + (4t^2 - 2)\,x \\ &= \big(\ddot{Y} + 4t\dot{Y} + (4t^2 + 2)Y\big)e^{t^2} - 4t\big(\dot{Y} + 2tY\big)e^{t^2} + (4t^2 - 2)\,Ye^{t^2} \\ &= \ddot{Y}\,e^{t^2}, \end{aligned}$$

   so that $Y(t) = Ct + D$. Hence the general solution is $x(t) = (Ct + D)e^{t^2}$, where C and D are arbitrary constants.

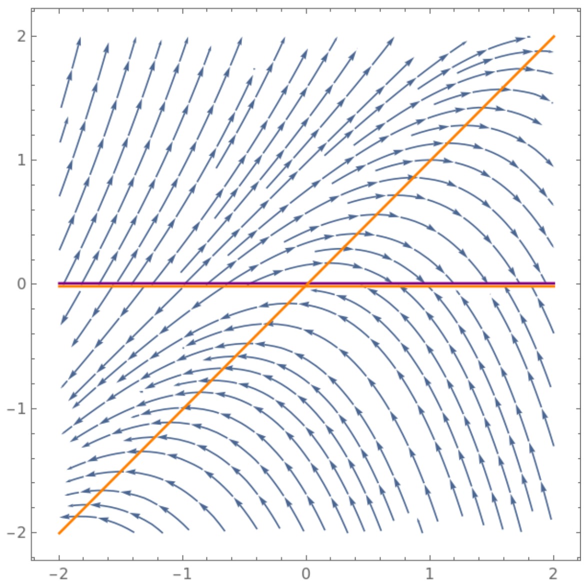
4. For $y \neq 0$ we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(y-x)y}{y} = y - x,$$

and this linear ODE $y' - y = -x$ has the general solution $y(x) = x + 1 + Ce^x$. This family of curves is easy to sketch; for $C = 0$ we get the straight line $y = x + 1$, and for other values of C an exponential gets added or subtracted to that; here is a figure showing these curves for $C \in \{-6, -5.5, -5, \dots, 5, 5.5, 6\}$:



Taking the nullclines (the lines $y = 0$ and $y = x$) and the signs of $\dot{x}$ and $\dot{y}$ into account, we add direction arrows to these curves in order to get the phase portrait shown below (note that every point on the x-axis is an equilibrium point):



5. The matrix A has a double eigenvalue $\lambda = 2$ with eigenvector $\binom{2}{1}$. This leads to the Jordan decomposition $A = MJM^{-1}$, where $M = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$, which then gives
$$e^{At} = M\exp(Jt)M^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e^{2t} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix} e^{2t}.$$

6. Obviously V is positive definite, and moreover $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y + xy - x^3 + x^4) + 2y(x - x^2 - y^3 + 3y^5) = -2x^4(1-x) - 2y^4(1-3y^2)$ is negative definite in the region
$$\Omega = \left\{(x, y) : x < 1, \, |y| < \tfrac{1}{\sqrt{3}}\right\},$$
so V is a strong Liapunov function in Ω. Hence, by Liapunov's theorem, the origin is asymptotically stable.

According to theory, the sublevel set $B_r = \{(x, y) : V(x, y) = x^2 + y^2 < r^2\}$ is a domain of stability for any $r > 0$ such that the closure of B_r is contained in the open set Ω, i.e., for $0 < r < \frac{1}{\sqrt{3}}$. This implies that also the union of all these B_r, i.e., the disk $N = B_{1/\sqrt{3}} = \{(x, y) : x^2 + y^2 < \frac{1}{3}\}$, is a domain of stability.

Phase portrait, including nullclines and the regions Ω and N: