Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

# TATA71 Ordinära differentialekvationer och dynamiska system

## Tentamen 2025-08-29 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3, 4, 5\}$ you need at least $n$ passed problems and at least $3n - 1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Consider the following model of a fish population $x(t)$ subjected to fishing:
$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \frac{ax}{b + x} \qquad (r, K, a, b > 0).$$
Interpret the terms and parameters on the right-hand side, and show how to rescale the variables $x$ and $t$ to obtain an ODE of the form
$$\frac{dy}{d\tau} = y(1 - y) + \begin{bmatrix} \text{something involving two} \\ \text{dimensionless parameters} \end{bmatrix}.$$
Please state clearly how the new variables and parameters are defined in terms of the original ones.

2. Find the general solution to the linear system
$$\dot{x} = -x - 2y, \qquad \dot{y} = 2x - y,$$
and draw the phase portrait.

3. Use linearization to classify the equilibrium points of the system
$$\dot{x} = y^3 - x, \qquad \dot{y} = y(x - 1),$$
and sketch the phase portrait.

4. Show that the origin is an asymptotically stable equilibrium of the system
$$\dot{x} = -y, \qquad \dot{y} = x - (1 - x^2)y^3,$$
and determine a domain of stability. (Hint: Try a very common Liapunov function.)

5. Given that $x(t) = e^t$ is a solution, use reduction of order to find the general solution to the ODE
$$\ddot{x}(t) - e^{-2t}\dot{x}(t) + \left(e^{-2t} - 1\right)x(t) = 0.$$

6. For a dynamical system with flow $\varphi_t$, define what it means for a point $\mathbf{y}$ to be an $\omega$-limit point of a point $\mathbf{x}$.

# Solutions for TATA71 2025-08-29

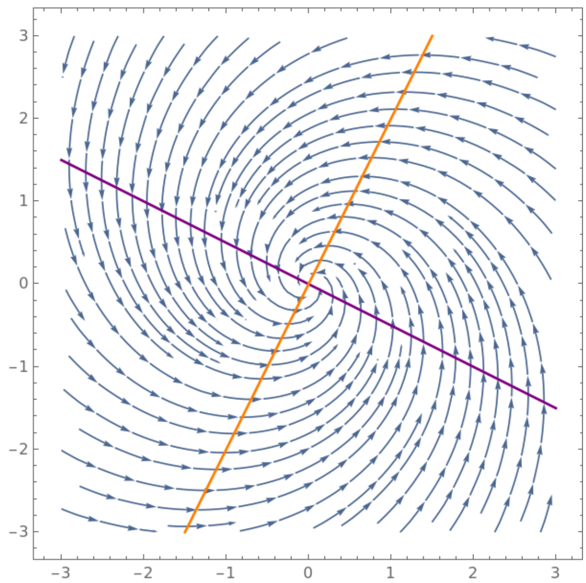
1. In the absence of fishing, the fish population would grow according to the well-known logistic equation $\dot{x} = rx\left(1 - \frac{x}{K}\right)$, where $r$ is the intrinsic growth rate per capita and $K$ is the carrying capacity of the environment. The term $\frac{ax}{b+x}$ represents the rate of removal of fish (due to fishing). This fishing rate increases with the fish population size $x$, but not without bounds; instead there is a *saturation* effect, where the fishing rate tends to $a$ as $x \to \infty$. So the parameter $a$ is the maximal fishing rate, while $b$ is the value of $x$ where the fishing rate equals $a/2$ (half the maximal rate).

   Changing to dimensionless variables $y = x/K$ and $\tau = rt$ turns the ODE into
   $$\frac{dy}{d\tau} = y(1-y) - \frac{a}{rK} \cdot \frac{y}{\frac{b}{K} + y} = y(1-y) - \frac{\alpha y}{\beta + y},$$
   where $\alpha = \frac{a}{rK}$ and $\beta = \frac{b}{K}$ are dimensionless parameters.

2. There are several ways of solving this system. For example, we can eliminate $y$ to get the single second-order ODE $\ddot{x} + 2\dot{x} + 5x = 0$, with the general solution $x(t) = e^{-t}(A\cos 2t + B\sin 2t)$, and from that compute $y(t) = -\frac{1}{2}(x + \dot{x}) = e^{-t}(A\sin 2t - B\cos 2t)$.

   **Answer.** $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ae^{-t}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + Be^{-t}\begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix}$ $(A, B \in \mathbf{R}$ arbitrary).

   The phase portrait is a stable focus, drawn here with the nullclines in purple ($x + 2y = 0$, for $x$) and orange ($2x - y = 0$, for $y$):

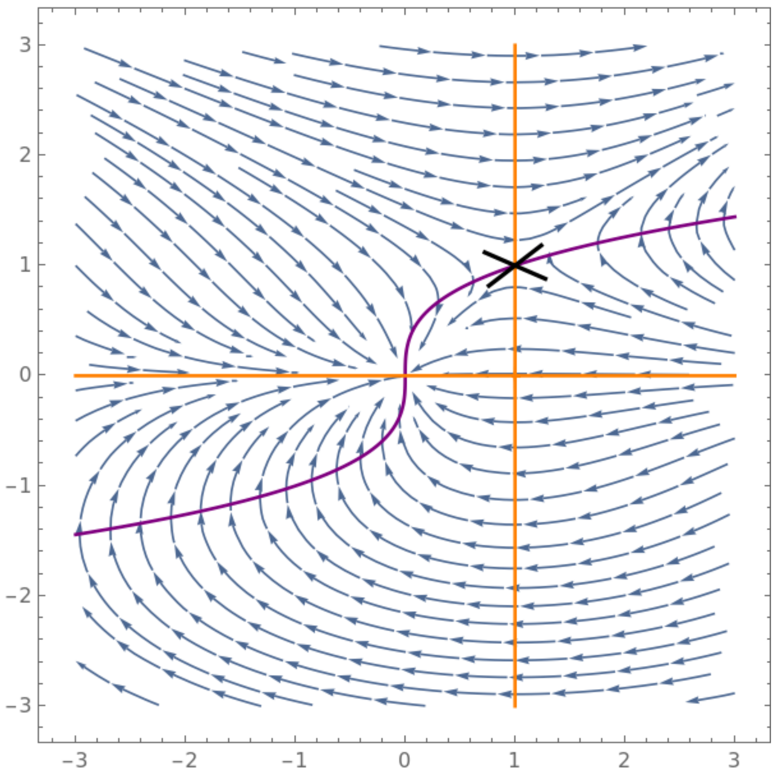


3. The equilibrium points are $(0,0)$ and $(1,1)$. Jacobian matrix:

$$J(x,y) = \begin{pmatrix} -1 & 3y^2 \\ y & x-1 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix}.$$

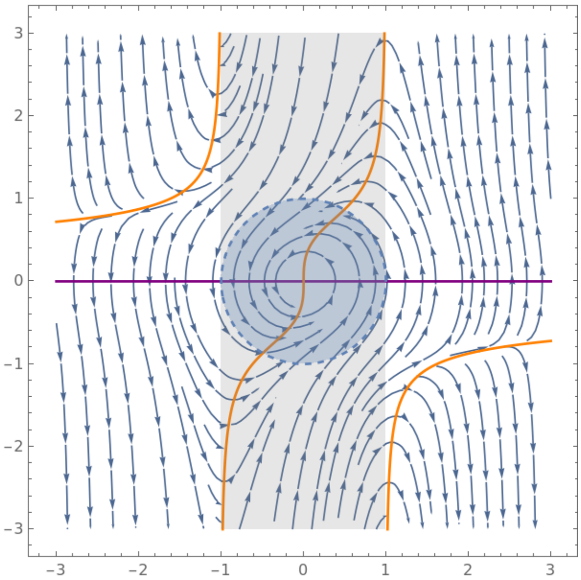
So $(0,0)$ is a stable star node, since the Jacobian there is a negative multiple of the identity matrix, and $(1,1)$ is a saddle, since the Jacobian there has negative determinant.

The phase portrait is shown below, including the nullclines and also indicating the principal directions at the saddle, given by the eigenvectors $(\lambda_1, 1)$ and $(\lambda_2, 1)$ corresponding to the eigenvalues $\lambda_1 = \frac{1}{2}(-1-\sqrt{13}) < 0$ and $\lambda_2 = \frac{1}{2}(-1+\sqrt{13}) > 0$, respectively.



4. $V(x,y) = x^2 + y^2$ is a weak Liapunov function for this system in the open strip $\Omega = \{(x,y) : -1 < x < 1\}$, since $V$ is positive definite and $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y) + 2y(x - (1-x^2)y^3) = -2(1-x^2)y^4$ satisfies $\dot{V} \le 0$ in $\Omega$. The set of points in $\Omega$ where $\dot{V} = 0$ is the segment $C = \{(x,0) : -1 < x < 1\}$. At those points, the vector field on the right-hand side of the system equals $(0,x)$, so it's transversal to $C$ except at the origin. According to LaSalle's theorem, the origin is therefore asymptotically stable. The usual arguments (not repeated here this time) show that the unit disk $x^2 + y^2 < 1$ is a domain of stability.

   **Remark.** A computer-drawn phase portrait (with $\Omega$ in gray and the unit disk in blue) shows that this is not the best possible domain of stability, but it's what we get from this Liapunov function:




5. With $x(t) = e^t Y(t)$ we have $\dot{x} = e^t(\dot{Y} + Y)$ and $\ddot{x} = e^t(\ddot{Y} + 2\dot{Y} + Y)$, so that the ODE becomes $0 = \ddot{x} - e^{-2t}\dot{x} + (e^{-2t} - 1)x = e^t[(\ddot{Y} + 2\dot{Y} + Y) - e^{-2t}(\dot{Y} + Y) + (e^{-2t} - 1)Y] = e^t[\ddot{Y} + (2 - e^{-2t})\dot{Y}] = e^t[\dot{y} + (2 - e^{-2t})y]$ where $y(t) = \dot{Y}(t)$. Using the integrating factor $\exp\left(\int (2 - e^{-2t})\,dt\right) = \exp\left(2t + \frac{1}{2}e^{-2t}\right)$ we find $y(t) = A\exp\left(-2t - \frac{1}{2}e^{-2t}\right) = Ae^{-2t}\exp\left(-\frac{1}{2}e^{-2t}\right)$, so that $Y(t) = \int y(t)\,dt = A\exp\left(-\frac{1}{2}e^{-2t}\right) + B$, from which we get the general solution $x(t) = e^t Y(t)$.

   **Answer.** $x(t) = A\exp\left(t - \frac{1}{2}e^{-2t}\right) + Be^t$.

6. It means that there is a sequence of times $t_n \to +\infty$ such that $\varphi_{t_n}(\mathbf{x}) \to \mathbf{y}$ as $n \to \infty$.