Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2026-01-15 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Draw the phase portrait for the one-dimensional dynamical system $\dot{x} = x^2 - 1$, and compute explicitly the solution $x(t)$ satisfying the initial condition $x(0) = -2$. (Also state the maximal time interval where this solution is defined.)

2. Calculate the general solution to the linear system $\dot{x} = x + y$, $\dot{y} = 2x$, and draw the phase portrait.

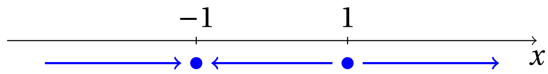
3. Use linearization to classify the equilibrium points of the system
$$\dot{x} = 2(y - x), \qquad \dot{y} = y - 2x + x^2,$$
and sketch the phase portrait.

4. (a) Define what it means for an equilibrium point $\mathbf{x}^*$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ to be **stable** (in the sense of Liapunov).
   (b) For the system $(\dot{x}, \dot{y}) = (-xy^2 - y^3, xy^2 - y^3)$, show that $(0,0)$ is a stable equilibrium. (Hint: Try a commonly used Liapunov function.)
   (c) For the same system, determine whether or not $(0,0)$ is *asymptotically* stable.

5. Find the general solution of the second-order ODE
$$\ddot{x} - 3\dot{x} + 2x = \frac{e^{2t}}{1+e^t}.$$

6. Consider the system $(\dot{x}, \dot{y}) = (y, x^2)$, with flow φ_t. For which points (x, y) is it true that $\varphi_t(x, y) \to (0,0)$ as $t \to +\infty$?

Solutions for TATA71 2026-01-15

1. Phase portrait: 

   To compute $x(t)$, we can for example use separation of variables and partial fractions. Since $x(0) = -2$, the solution must stay in the interval $x < -1$ for as long as it is defined. (We can see already in the phase portrait that it must be defined for all $t > 0$, and approach the equilibrium -1 as $t \to +\infty$. However, the solution formula below shows that actually $x(t) \to -\infty$ in finite negative time, namely when e^{2t} reaches the value $\frac{1}{3}$.) In the interval $x < 1$, both $x-1$ and $x+1$ are negative, so $\frac{x-1}{x+1}$ is positive:

$$t = \int_0^t dt = \int_{-2}^{x(t)} \frac{dx}{x^2-1} = \int_{-2}^{x(t)} \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$
$$= \left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_{-2}^{x(t)} = \frac{1}{2}\ln\left|\frac{x(t)-1}{x(t)+1}\right| - \frac{1}{2}\ln\left|\frac{-2-1}{-2+1}\right|$$
$$= \frac{1}{2}\ln\frac{x(t)-1}{x(t)+1} - \frac{1}{2}\ln 3,$$

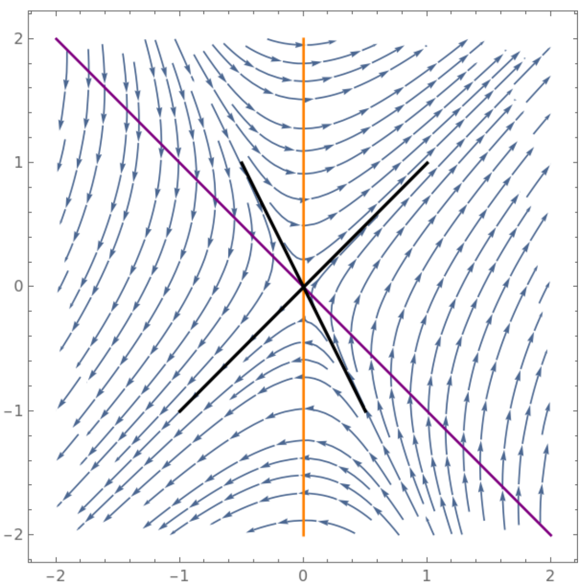
   and from this we can solve for $x(t)$.

   Answer. $x(t) = \dfrac{-1-3e^{2t}}{-1+3e^{2t}}$, for $t > -\frac{1}{2}\ln 3$.

2. Using the eigenvalues and eigenvectors of the system's matrix $\left(\begin{smallmatrix}1 & 1\\ 2 & 0\end{smallmatrix}\right)$, we can write down the general solution immediately.

   Answer. $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = Ae^{2t}\begin{pmatrix}1\\ 1\end{pmatrix} + Be^{-t}\begin{pmatrix}-1\\ 2\end{pmatrix}$ $\quad$ ($A, B \in \mathbf{R}$ arbitrary).

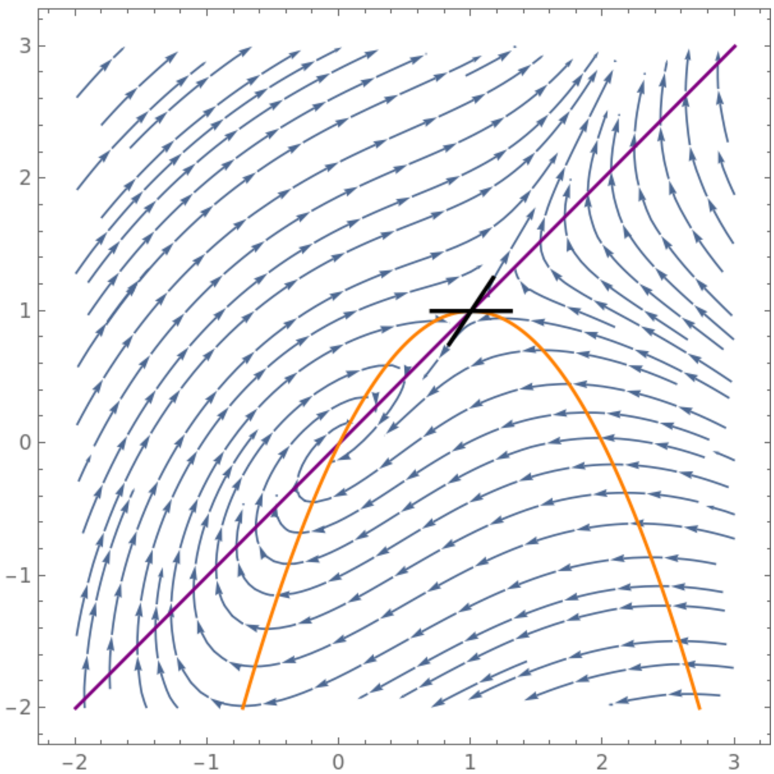
   The phase portrait is a saddle, drawn here with the nullclines in purple ($y = -x$, for x) and orange ($x = 0$, for y), and with the principal directions indicated by black line segments:



3. The equilibrium points are $(0,0)$ and $(1,1)$. Jacobian matrix:

$$J(x,y) = \begin{pmatrix} -2 & 2 \\ 2x-2 & 1 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} -2 & 2 \\ -2 & 1 \end{pmatrix}.$$

So $(1,1)$ is a saddle; the eigenvalues -2 and 1 can be read off from the diagonal of the triangular matrix $J(1,1)$, and the corresponding eigenvectors are $\binom{1}{0}$ and $\binom{2}{3}$. And $(0,0)$ is a stable focus, by the trace–determinant criterion, since $\beta = \operatorname{tr} J(0,0) = -1$ and $\gamma = \det J(0,0) = 2$ satisfy $\beta < 0$ and $(\beta/2)^2 < \gamma$. The phase portrait is shown below, including the nullclines and the principal directions at the saddle:



4. (a) By definition, the equilibrium point $\mathbf{x}^*$ is stable iff for every neighbourhood U of $\mathbf{x}^*$ there is some neighbourhood V of $\mathbf{x}^*$ such that all solutions starting in V remain in U for all later times.

   (b) $V(x,y) = x^2 + y^2$ is a weak Liapunov function for this system (in the whole plane $\mathbf{R}^2$), since V is positive definite and $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-xy^2 - y^3) + 2y(xy^2 - y^3) = -2y^2(x^2 + y^2)$ satisfies $\dot{V} \le 0$ everywhere. So $(0,0)$ is stable, by the weak version of Liapunov's theorem.

   (c) No, $(0,0)$ is not asymptotically stable, for the simple reason that every point on the x-axis is an equilibrium point, causing every neighbourhood of $(0,0)$ to contain solutions which don't converge to $(0,0)$ as $t \to \infty$.

5. The characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$, so the general solution of the homogeneous equation is $x_{\text{hom}}(t) = Ae^t + Be^{2t}$. With $(x_1, x_2) = (x, \dot{x})$ the ODE can be written as a system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2t}(1+e^t)^{-1} \end{pmatrix}.$$

Since e^t and e^{2t} form a basis for the solution space of the homogeneous equation, we can obtain a fundamental matrix for the system as follows:

$$\Phi(t) = \begin{pmatrix} e^t & e^{2t} \\ \frac{d}{dt}(e^t) & \frac{d}{dt}(e^{2t}) \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}.$$

Letting $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$ leads in the usual way to

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \Phi(t)^{-1} \begin{pmatrix} 0 \\ e^{2t}(1+e^t)^{-1} \end{pmatrix} = \begin{pmatrix} 2e^{-t} & -e^{-t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ e^{2t}(1+e^t)^{-1} \end{pmatrix} = \frac{1}{1+e^t} \begin{pmatrix} -e^t \\ 1 \end{pmatrix},$$

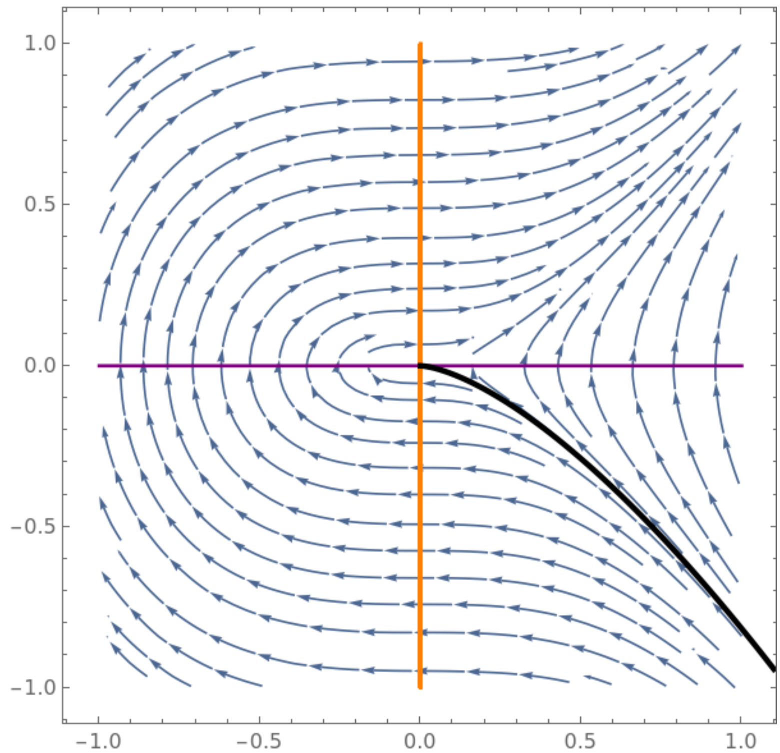
so that $y_1(t) = \int \frac{-e^t\,dt}{1+e^t} = -\ln(1+e^t) + A$ and $y_2(t) = \int \frac{dt}{1+e^t} = \int \frac{e^{-t}\,dt}{e^{-t}+1} = -\ln(1+e^{-t}) + B$. Finally we find the sought solution $x(t) = x_1(t) = e^t y_1(t) + e^{2t} y_2(t)$ from the first component in the matrix product $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$.

Answer. The general solution is

$$x(t) = -e^t \ln(1+e^t) - e^{2t} \ln(1+e^{-t}) + Ae^t + Be^{2t},$$

where A and B are arbitrary constants.

6. The function $H(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3$ is a constant of motion, as can be seen by noticing that the system has the Hamiltonian form $(\dot{x}, \dot{y}) = (H_y, -H_x)$, or via the usual procedure of eliminating the time variable: $dy/dx = \dot{y}/\dot{x} = x^2/y$, leading to $\int y\,dy = \int x^2\,dx$. So the trajectories of the system follow the level curves of H, and by considering the phase portrait we see that the only solutions tending to the origin as $t \to \infty$ are those on the lower half (drawn in black below) of the level curve $H(x,y) = 0 \iff y = \pm\sqrt{\frac{2}{3}x^3}$, including the equilibrium solution $(x(t), y(t)) = (0,0)$ itself:



Answer. The points on the curve $y = -\sqrt{\frac{2}{3}x^3}$, $x \geq 0$.