Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2026-03-18 kl. 14.00–19.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Find $x(t)$ and $y(t)$ such that

   $$\dot{x} = x - 6y, \quad \dot{y} = 6x + y, \quad x(0) = 1, \quad y(0) = 0.$$

   Sketch the solution curve $(x(t), y(t))$ in the xy-plane.

2. Use linearization to classify the equilibrium points of the system

   $$\dot{x} = 2y - x - x^2, \qquad \dot{y} = x(xy - 1),$$

   and sketch the phase portrait.

3. Find the general solution $x(t)$ of the ODE

   $$\tfrac{1}{2}(t^2+1)^2\ddot{x} - t(t^2+1)\dot{x} + (t^2-1)x = 0.$$

   (Hint: $x(t) = t^2 + 1$ is a solution.)

4. Consider the logistic model with a constant carrying capacity $K > 0$ (as usual), but with a time-dependent growth rate given by some continuous function $r(t) > 0$:

   $$\frac{dx}{dt} = r(t)\,x\left(1 - \frac{x}{K}\right).$$

   Calculate the solution $x(t)$, expressed in terms of the initial value $x(0) = x_0$ and the function $R(t) = \int_0^t r(\tau)\,d\tau$.

5. Rewrite the system

   $$\dot{x} = -y + (1 - x^2 - y^2)x, \qquad \dot{y} = x + (1 - x^2 - y^2)y$$

   in polar coordinates (r, θ), and use this to determine the ω-limit set $L_\omega(x_0, y_0)$ of each point $(x_0, y_0) \in \mathbf{R}^2$.

6. Show that the origin is an asymptotically stable equilibrium point for the system

   $$\dot{x} = y, \qquad \dot{y} = -x^3 - y^3 + y^5,$$

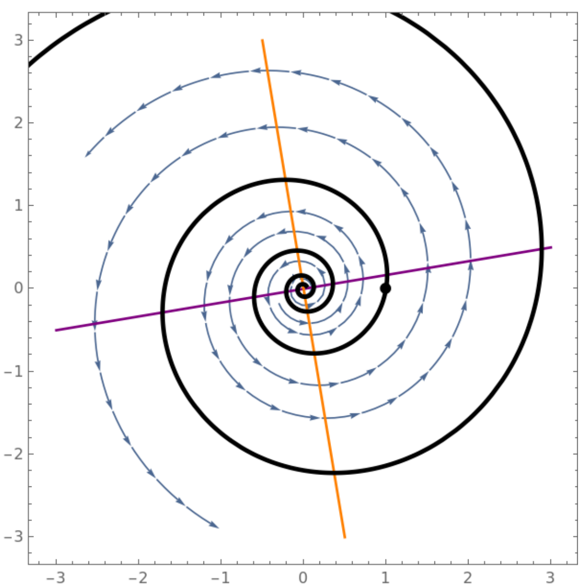
   and determine a domain of stability. (Hint: $V(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2$.)

Solutions for TATA71 2026-03-18

1. This is a canonical linear system $\frac{d}{dt}\binom{x}{y} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\binom{x}{y}$ with $a = 1$ and $b = 6$, so the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} t\right)\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{t}\begin{pmatrix} \cos 6t \\ \sin 6t \end{pmatrix}.$$

The phase portrait is an unstable focus, with solution curves spiralling outwards counterclockwise, and the particular curve asked for here is just the spiral which passes through the point $(1, 0)$. In the figure below, it is drawn in black, together with the nullclines and a couple of other spirals. For each lap around the origin, the radius expands by a factor of 3, roughly speaking. (The time it takes to go one lap is $2\pi/6 = \pi/3$, and then the radius expands by the factor $e^{\pi/3} \approx e^{1.05} \approx 3$.)



2. We have $(\dot{x}, \dot{y}) = (0, 0) \iff 0 = 2y - x - x^2$ and $(x = 0$ or $y = \frac{1}{x}) \iff (x, y) = (0, 0)$ or $(y = \frac{1}{x}$ and $0 = \frac{2}{x} - x - x^2)$. The last equation is equivalent to $0 = x^3 + x^2 - 2 = (x-1)(x^2 + 2x + 2) = (x-1)\big((x+1)^2 + 1\big)$, so $x = 1$ is the only real solution, with corresponding $y = \frac{1}{x} = 1$. Thus the equilibrium points are $(0, 0)$ and $(1, 1)$.

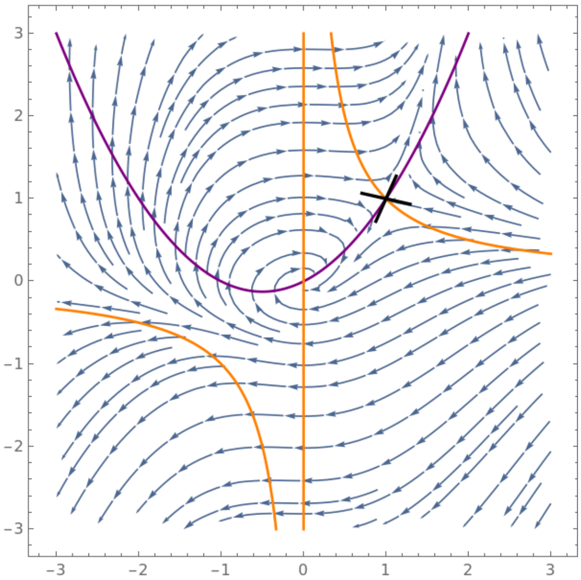
(One may also obtain this by drawing the nullclines and locating their intersections graphically, as in the picture on the next page: $\dot{x} = 0$ on the parabola $y = \frac{1}{2}x(x+1) = \frac{1}{2}\big((x + \frac{1}{2})^2 - \frac{1}{4}\big)$, and $\dot{y} = 0$ on the y-axis and on the curve $y = \frac{1}{x}$.)

Jacobian matrix:

$$J(x,y) = \begin{pmatrix} -1-2x & 2 \\ 2xy-1 & x^2 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}.$$

So $(0,0)$ is a stable focus, by the trace–determinant criterion, since $\beta = \operatorname{tr} J(0,0) = -1$ and $\gamma = \det J(0,0) = 2$ satisfy $\beta < 0$ and $(\beta/2)^2 < \gamma$. And $(1,1)$ is a saddle, since $\gamma = \det J(1,1) = -5 < 0$. (The eigenvalues are $-1 \pm \sqrt{6}$ with corresponding eigenvectors $\binom{-2\pm\sqrt{6}}{1}$.)

The phase portrait is shown below, including the nullclines and the principal directions at the saddle:



3. By setting $x(t) = (t^2+1)\,Y(t)$, we reduce the order and obtain a first-order ODE for $y(t) = \dot{Y}(t)$:

$$\begin{aligned}
0 &= \tfrac{1}{2}(t^2+1)^2\ddot{x} - t(t^2+1)\dot{x} + (t^2-1)x \\
&= \tfrac{1}{2}(t^2+1)^2\big(2Y + 4t\dot{Y} + (t^2+1)\ddot{Y}\big) - t(t^2+1)\big(2tY + (t^2+1)\dot{Y}\big) + (t^2-1)(t^2+1)Y \\
&= \tfrac{1}{2}(t^2+1)^3\ddot{Y} + \big(2t(t^2+1)^2 - t(t^2+1)^2\big)\dot{Y} + \big((t^2+1)^2 - 2t^2(t^2+1) + (t^4-1)\big)Y \\
&= \tfrac{1}{2}(t^2+1)^3\ddot{Y} + t(t^2+1)^2\dot{Y} + 0\,Y \\
&= \tfrac{1}{2}(t^2+1)^2\big((t^2+1)\dot{y} + 2ty\big) \quad \Longleftrightarrow \quad 0 = \tfrac{d}{dt}\big((t^2+1)y\big) \quad \Longleftrightarrow \quad y(t) = \tfrac{C}{t^2+1}.
\end{aligned}$$

Integration gives $Y(t) = \int y(t)\,dt = C \arctan t + D$, where C and D are arbitrary constants.

Answer. $x(t) = (t^2+1)\,(C\arctan t + D)$.

4. The ODE is separable. There are constant solutions $x(t) = 0$ and $x(t) = K$ corresponding to $x_0 = 0$ and $x_0 = K$. The other solutions, with $x_0 \notin \{0, K\}$, are given by

$$\int \frac{dx}{x\left(1 - \frac{x}{K}\right)} = \int r(t)\, dt.$$

To take the initial condition into account, we can write

$$\int_{x_0}^{x(t)} \left(\frac{1}{x} + \frac{\frac{1}{K}}{1 - \frac{x}{K}} \right) dx = \int_0^t r(\tau)\, d\tau,$$

or in other words

$$R(t) = \ln \left| \frac{x(t)}{1 - \frac{x(t)}{K}} \right| - \ln \left| \frac{x_0}{1 - \frac{x_0}{K}} \right| = \ln \left| \frac{\frac{K x(t)}{K - x(t)}}{\frac{K x_0}{K - x_0}} \right|.$$

The solution $x(t)$ must stay in the same interval $(-\infty, 0)$, $(0, K)$ or (K, ∞) where the initial value x_0 lies, and this makes the quotient positive in the righmost expression above, so that we can remove the absolute value signs. Thus,

$$e^{R(t)} = \frac{\frac{K x(t)}{K - x(t)}}{\frac{K x_0}{K - x_0}} \iff \frac{x(t)}{K - x(t)} = \frac{x_0\, e^{R(t)}}{K - x_0}$$

$$\iff x(t) = \frac{K \frac{x_0\, e^{R(t)}}{K - x_0}}{1 + \frac{x_0\, e^{R(t)}}{K - x_0}} = \frac{K x_0\, e^{R(t)}}{K + x_0 (e^{R(t)} - 1)}.$$

A direct verification shows that this last formula gives the correct solution also for $x_0 = 0$ and for $x_0 = K$, so it holds in all cases. The solution blows up if the denominator $K + x_0(e^{R(t)} - 1)$ becomes zero; whether this happens (and when) depends on the function $r(t)$ and the value of x_0. In any case, the solution is defined in the maximal time interval around $t = 0$ where the denominator stays nonzero.

Answer. $x(t) = \dfrac{K x_0\, e^{R(t)}}{K + x_0 (e^{R(t)} - 1)}$.

5. The usual formulas give

$$\dot r = \frac{x\dot x + y\dot y}{r} = \frac{x\big(-y+(1-r^2)x\big)+y\big(x+(1-r^2)y\big)}{r} = r(1-r^2)$$

and

$$\dot\theta = \frac{x\dot y - y\dot x}{r^2} = \frac{x\big(x+(1-r^2)y\big)-y\big(-y+(1-r^2)x\big)}{r^2} = 1.$$

Since $(x,y) = (0,0)$ is an equilibrium point, we have $L_\omega(0,0) = \{(0,0)\}$. The one-dimensional phase portrait for $\dot r = r(1-r^2)$ shows that all other solutions (with $r > 0$) approach the stable limit cycle given by $r = 1$ and $\dot\theta = 1$, i.e., the unit circle (traversed at constant speed). Thus, $L_\omega(x_0, y_0) = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ for every point $(x_0, y_0) \neq (0,0)$.

6. The function $V(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2$ is clearly positive definite, and moreover

$$\dot V = x^3\dot x + y\dot y = x^3 y + y(-x^3 - y^3 + y^5) = -y^4(1-y^2)$$

satisfies $\dot V \le 0$ in the open strip $\Omega = \{(x,y) \in \mathbf{R}^2 : -1 < y < 1\}$, so V is a weak Liapunov function in Ω. The set in Ω where $\dot V = 0$ is the line $y = 0$, where the vector field $(y, -x^3 - y^3 + y^5)$ reduces to $(0, -x^3)$, which is transversal to the line $y = 0$ at all points except for the equilibrium point $(0,0)$. So the hypotheses of LaSalle's theorem are satisfied, which shows that the origin is asymptotically stable.

For each k with $0 < k < \frac{1}{2}$, the sublevel set $B(k) = \{(x,y) \in \mathbf{R}^2 : V(x,y) \le k\}$ is a closed topological ball contained in Ω, so that (according to the usual recipe) its interior $N(k) = \{(x,y) \in \mathbf{R}^2 : V(x,y) < k\}$ is a domain of stability. The union of these sets (the dark blue region in the phase portrait shown below) is then also a domain of stability:

$$\bigcup_{0<k<\frac{1}{2}} N(k) = N\big(\tfrac{1}{2}\big) = \Big\{(x,y) \in \mathbf{R}^2 : \tfrac{1}{4}x^4 + \tfrac{1}{2}y^2 < \tfrac{1}{2}\Big\}.$$