Hand-in Exercises TATA74 1 Fall 2025: Curves

First of all the exercises are to be solved individually: **it is your examination!**

The exercises to be done by each of you are parametrised by $(M_1 - M_2, D_1 - D_2, Y_1 - Y_2)$, which are the moth, day and year of your birthday, but if someone is born year 2000, for this course the student is born 1998. Mine is (1-2, 1-5, 6-3). Some one born March 3 1990 has coordinates (0-3, 0-3, 9-0).

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 1, 2, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \text{ mod} 3$$

where $l = 1$ for an exercise type **a)**, $l = -1$ for an exercise type **b)**, and $l = 0$ for type **c)**.

So I should solve exercises $a.3$ and $b.1$ in Exercise 2, and 1 and 2 in Exercise 6.

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

Recall that you can always calculate numerically the length of an arc using, for instance, MATLAB. Below you find the command

Integration in MATLAB

`q = integral(fun,xmin,xmax)` numerically integrates function `fun` from `xmin` to `xmax` using global adaptive quadrature and default error tolerances.

But remember to integrate numerically we must tell MATLAB that a function is taken numerically by writing before th eexpression of the function "@(x)".

Example:

```
fun = @(x)sin(x);
q = integral(fun,0,pi/2)
```

```
>implicitplot([seq(x*cos(Pi*s/20) + y*sin(Pi*s/20) = sin(Pi*s/20)*cos(Pi*s/20), s = [0, 1,
```

Exercise 1 *Calculate the curvature ,and torsion if applicable, at a generic point of the parametrised curves as well as the length of the following arcs of curves (for plane curves the curvature is signed):*

a.1 $\gamma(t) = (a\frac{\cos(t)}{(\sin(t))^2+1}, a\frac{\cos(t)\sin(t)}{(\sin(t))^2+1})$, *with* $t \in [0, 2\pi]$. $a > 0$ *constant: The lemniscata.*

a.2 $\gamma(t) = (a\frac{t^2-1}{t^2+1}, a\frac{t(t^2-1)}{t^2+1})$, *with* $t \in [-2, 2]$. $a > 0$ *constant: the strophoid.*

a.3 $\gamma(t) = (at - a\sin(t), a - a\cos(t))$, *with* $t \in [0, 2\pi]$, $a > 0$ *constant. An arc of cycloid.*

b.1 The plane curve $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = 2a(\cos\phi + 1)$, $\phi \in [0, 2\pi]$. *Remember if* (r, ϕ) *are the polar coordinates of a point on the plane, its Cartesian coordinates are* $(x = r\cos(\phi),\ y = r\sin(\phi))$. *This curve is called the cardioid.*

b.2 The plane curve $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = 2a\cos(3\phi) + 2$ *with* $\phi \in [0, 2\pi]$*: a limacon (notice that the cardioid is a particular limacon).*

b.3 $\gamma(t) = (\frac{(1+t)^{3/2}}{3}, \frac{(1-t)^{3/2}}{3}, \frac{t}{\sqrt{2}})$, *with* $t \in [-1, 1]$..

c.1 $\gamma(t) = (a\cosh(t)), a\sinh(t), bt)$, $t \in [-5, 5]$, *with* a, b *positive constants.*

c.2 $\gamma(t) = (a(1 + \cos(t)), a\sin(t), 2a\sin(t/2))$, $t \in [-2\pi, 2\pi]$ *with* a *a positive constant. This is Viviani's curve: the intersection of the sphere* $x^2 + y^2 + z^2 = 4a^2$ *with the cylinder* $(x - a)^2 + y^2 = a^2$.

c.3 $\gamma(t) = (3t - t^3, 3t^2, 3t + t^3)$, *with* $t \in [-2, 2]$.

d.1 $\gamma(t) = (\sqrt{3}t - \sin t, 2\cos t, t + \sqrt{3}\sin t)$, $t \in [-\pi, \pi]$

d.2 The graph of the function $x = a\cosh(t/a)$, $t \in [-5, 5]$ *with* $a > 0$ *constant: a catenary.*

d.3 $\gamma(t) = (a\sin(t), a\cos(t) + \log(\tan(\frac{t}{2})))$, $t \in (0, \pi)$. *A tractrix.*

Exercise 2 *Consider the unit-speed curve* $\gamma(s)$ *with Frenet-trihedron* $\boldsymbol{t}$, $\boldsymbol{n}$ *and* $\boldsymbol{b}$, *curvature* $\kappa \neq 0$ *and torsion* τ. *Show that*

a.1 $\frac{[\mathbf{n}, \mathbf{n}', \mathbf{n}'']}{|\mathbf{n}'|^2} = \frac{(\frac{\kappa}{\tau})'}{(\frac{\kappa}{\tau})^2 + 1}$

a.2 $[\mathbf{b}', \mathbf{b}'', \mathbf{b}'''] = \tau^5(\frac{\kappa}{\tau})'$. *Notation:* $(\frac{\kappa}{\tau})'$ *is the derivative of* $(\frac{\kappa}{\tau})$.

a.3 $[\mathbf{t}', \mathbf{t}'', \mathbf{t}'''] = \kappa^5(\frac{\tau}{\kappa})'$. *Notation:* $(\frac{\tau}{\kappa})'$ *s the derivative of* $\frac{\tau}{\kappa}$.

b.1 Show that if $\widehat{\kappa}$ *and* $\widehat{\tau}$ *are the curvature and torsion of the spherical curve* $\widehat{\gamma}(s) = \mathbf{t}(s)$ *then*

$$\widehat{\kappa} = \sqrt{1 + (\frac{\tau}{\kappa})^2} \quad \widehat{\tau} = \frac{(\frac{\tau}{\kappa})'}{\kappa(1 + (\frac{\tau}{\kappa})^2)}$$

b.2 Let $\gamma : I \to \mathbb{R}^3$ *be a unit-speed curve with nowhere vanishing torsion* τ. *Consider the curve* $\overline{\gamma} = \int_{s_0}^{s} \mathbf{b}(s)ds$, *called the adjoint curve of* γ. *(***b** *is the binormal vector to* γ*). Show that if* γ *has constant curvature (resp. torsion), then* $\overline{\gamma}$ *has constant torsion (resp. curvature).*

b.3 Let $\gamma : I \to \mathbb{R}^3$ be a unit-speed curve with nowhere vanishing constant torsion τ. Calculate the curvature of $\widehat{\gamma}(s) = \frac{-\mathbf{n}}{\tau} + \int_{s_0}^{s} \mathbf{b}(s)ds$.

Exercise 3 *Consider $X = \{(x, y, z) \in \mathbb{R}^3, x^2 - x + z^3 = 0,\ x^3 - y^2 + z^3 = 0\}$. By considering a map $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by $F(x, y, z) = (F_1 = x^2 - x + z^3, F_2 = x^3 - y^2 + z^3)$ (notice that $(0, 0)$ is a regular value of F) show that there exists an open neighbourhood U of $P(1, 1, 0)$ such that the geometric locus $\mathcal{C} = X \cap U$ admits a regular parametrisation $\gamma : (1/2, 3/2) \to \mathbb{R}^3$ with $\gamma(1) = P$. Determine the tangent line and the torsion of γ at P.*

Exercise 4 *Consider the vector $\mathbf{u} = (0, 0, 1)$ and the plane H with equation $x_3 = 0$. Given a plane regular parametrised curve $\beta : I \to \mathbb{R}^3$ given by $\beta(t) = (x(t), y(t), 0)$ consider the parametrisation $\gamma : I \to \mathbb{R}^3$ given by $\gamma(t) = \beta(t) + t\mathbf{u}$. Show that $\gamma(t)$ is a regular parametrisation. Determine the curvature and torsion of γ at a point $P = \gamma(t)$ in terms of the curvature of β at the corresponding point $Q = \beta(t)$.*

Exercise 5 *a.1 Let $\gamma(s)$ a unit-speed curve s.t. $\tau(s) \neq 0$, for any value of s. The curve $\beta(s) = \int_{s_0}^{s} \mathbf{b} d\sigma$ is called the adjoint curve to γ. Show that if γ has constant curvature (resp. torsion), so has β constant torsion (resp. curvature).*

a.2 Determine the points on $\gamma(t) = (2/t, \ln(t), -t^2)$, $t > 0$, such that the binormal line to the curve at $\gamma(t)$ is parallel to the plane $x - y + 8z + 2 = 0$

a.3 Let $\gamma(s)$ be a unit-speed parametrised curve. Let $\mathbf{u}(s)$ be a unitary vector on the plane generated by $\mathbf{n}$ and $\mathbf{b}$. Let $\sigma(s)$ be the angle between $\mathbf{u}(s)$ and $\mathbf{n}(s)$. Consider the curve $\beta(s) = \gamma(s) + \lambda(s)\mathbf{u}$.

Show that if $\frac{d\sigma}{ds} = -\tau$, so the straight line between $\gamma(s_0)$ and $\beta(s_0)$ is in fact the tangent line to $\beta(s)$ at $\beta(s_0)$. For which $\lambda(s)$ does it occur $\frac{d\sigma}{ds} = -\tau$?

b.1 Consider a unit-speed parametrised curve $\gamma : I \to \mathbb{R}^3$ with no inflexion points. Show that if γ'', γ''' and $\gamma^{(iv)}$ are linearly independent then $\frac{\kappa(s)}{\tau(s)}$ is a constant.

b.2 Show that the curve $\gamma(t)$ with parametrisation $(16\cos(t)/9 - 32\sin(t)/9 - t/3, 16\cos(t)/9 + 4\sin(t)/9 + 8t/3, 28\cos(t)/9 + 16\sin(t)/9 - 4t/3)$ is a helix and determine its axis.

b.3 Let $\gamma(s)$ be a circular helix. Consider $\beta(s) = \gamma(s) + \mathbf{b}(s)$. Show that $\beta(s)$ is a helix. Notice that s is not the arc-length of $\beta(s)$

Exercise 6 *a.1 Determine a plane curve with $\overline{\kappa} = \frac{1}{bs}$, s arc-length, $b > 0$ a constant. Can you see that this curve is the logarithmic spiral?*

a.2 Determine a plane curve swith $\overline{\kappa} = -s$, s arc-length. This curve is called the clothoid.

a.3 *Determine a plane curve such that* $\overline{\kappa} = \sin(s)$, *s arc-length.*

Exercise 7 b.1 *Let* $\mathbf{F} : I \to \mathbb{R}^3$, $I = (-\pi, \pi)$ *be defined by* $\mathbf{F}(t) = (\sin t, \sin t \cos t, \cos^2 t)$. *Determine the unitary tangent vector* $\mathbf{t}$ *to a parametrization* $\gamma : I \to \mathbb{R}^3$ *whose torsion function* τ *is constant of value 2 and whose binormal vector* $\mathbf{b}(t) = \mathbf{F}(t)$. *(Observe that* $\mathbf{b}$ *determines* κ *and* $|\tau|$. *Is this true?).*

b.2 *Integrate the Frenet-Serret equations to show that, if the curvature and the torsion of a regular curve* $\gamma(t)$ *are* $\kappa \neq 0$ *and* $\tau = 1/a$ *(a a constant) , then* $\gamma(t) = a \int g(t) \times g'(t) dt$, *where* $g(t)$ *is a vectorial function satisfying that* $|g(t)| = 1$ *and* $[g, g', g''] \neq 0$.

b.3 *Using Exercise 2.a.1 show that the normal vector* $\mathbf{n}$ *of a unit-speed curve without inflexion points determine the curvature and torsion of the curve.*

Exercise 8 *Consider the family of curves defined by* $F(x, y) = a$. *We will consider that* $F_y = \frac{\partial F}{\partial y} \neq 0$

a.1 *Consider the family of curves given by* $G(x, y) = b$ *(again* $G_y = \frac{\partial G}{\partial y} \neq 0$*). Show that if the condition* $\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} = 0$ *is satisfied, then each curve in the first family is orthogonal to each curve of the second family at the intersection point.*

a.2 *Give the differential equation for the family of curves formed by those curves that intersect to each curve in the family* $F(x, y) = a$ *orthogonally.*

a.3 *Deterrmine the family of lines orthogonal to the circles tangent to the* x_1*-axis at the origin* O.

For the Exercises type b the curves are plane ones.

b.1 *Show that* $\kappa = \frac{|\frac{d^2y}{dx^2}|}{(1+(\frac{dy}{dx})^2)^{3/2}}$, *for a curve* $y = y(x)$.

b.2 *Show that* $\kappa = \frac{|F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2|}{(F_x+F_y)^{3/2}}$. *Give the equation for the inflexion points*

b.3 $\kappa = \frac{r^2+2(\frac{dr}{d\varphi})^2 - r\frac{d^2r}{d\varphi^2}}{(r^2+(\frac{dr}{d\varphi})^2)^{3/2}}$, *for a curve in polar form* $r = r(\varphi)$.

Exercise 9 *Determine the cubic Bézier curve* $B(t), 0 \le t \le 1$ *joining* $A(M_1.M_2, D_1.D_2)$ *and* $B(1.9, Y_1.Y_2)$ *with tangent at* $A = B(0)$ *making a* $\frac{\pi}{4}$*-angle with the* x_1*-axis and horizontal tangent at* $B = B(1)$. *We know also that* $B(1/2) = P(Y_1, Y_2)$. *Determine the curve* $B(t)$.

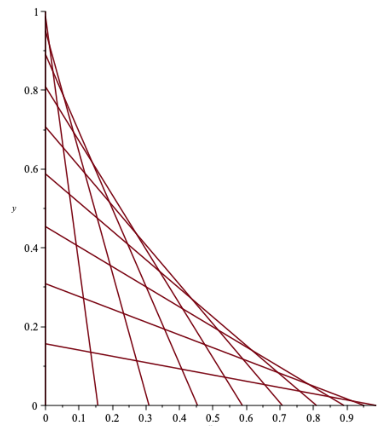
Exercise 10 a) *Let* $\gamma : [a, b] \to \mathbb{R}^2$ *be a parametrisation of a unit-speed simple closed, convex curve on the plane. Show that* $\mathbf{t}''$ *is parallel to* $\mathbf{t}$ *at at least four points of the curve.*

b) *Consider the cardioid with parametrisation* $r(\phi) = 1 - 2\sin(t)$, $t \in []0, 2\pi$. *Show that the curve has only two vertices. The condition of the curve to be convex is necessary.*

Exercise 11 *Consider the family of segments* $[A_t, B_t] = \{(1-t)A_t + tB_t, 0 \leq t \leq 1\}$ *of fixed length* Y_2 *with end points* $A_t(x_1(t), 0), B_t(0, x_2(t))$ *obtained by sliding the segment* $[A_0(Y_2, 0), B_0(0, 0)]$. *Consider the family of straight lines* $\{l_t\}_{t\in[0,2\pi]}$ *containing the segments above. Determine the envelope of the family of lines. Is it the envelope a regular curve? Calculate the length of the curve (you may use symmetries in the calculations) and the signed curvature at a regular point. Finally show that a rotation around the origin of angle* $\pi/2$ *is a symmetry of the curve.*

The picture of some segments has been made with MAPLE:

```
>implicitplot([seq(x*cos(Pi*s/20) + y*sin(Pi*s/20) = sin(Pi*s/20)*cos(Pi*s/20), s = [0, 1,
2, 3, 4, 5, 6, 7, 8, 9])], x = 0 .. 1, y = 1 .. 0);
```



Exercise 12 **a)** *Show that the equations for the envelope of the* ***uniparametric*** *family of plane curves given by* $F(x, y, a, b) = 0$, *where the parameters* a, b *satisfy the condition* $\varphi(a, b) = 0$ *are*

$$F(x, y, a, b) = 0, \quad \varphi(a, b) = 0, \quad det(\frac{\partial(F, \varphi)}{\partial(a, b)}) = 0$$

($\frac{\partial(F,\varphi)}{\partial(a,b)}$ *is the Jacobian of the function* $(F(\cdot, \cdot, a, b), \varphi(a, b))$)

b) *We know that the envelope of the family of straight lines* $ax + y + b = 0$ *is the circle with equation* $x^2 + y^2 = c^2$, *with* a, b *parameters and* c *a constant. Give the condition satisfied by* a *and* b *(the function* $\varphi(a, b) = 0$*).*

c) *Let* $\mathbf{x}(s)$, $s \in [0, l]$, *be a plane simple closed curve such that its curvature satisfies* $0 < \kappa(s) \leq c$, *with* c *a constant. Prove that* $l \geq \frac{2\pi}{c}$.

d) *Let* $\mathbf{x}(s)$ *be a plane closed curve with rotation index* I *such that its curvature satisfies* $0 < \kappa(s) \leq c$, *with* c *a constant. Prove that* $l \geq \frac{2\pi I}{c}$, *where* l *is the length of* $\mathbf{x}(s)$.