Hand-in Exercises TATA74 Surfaces 1: Tangent plane, I- and II-fundamental forms. Fall 2025

First of all the exercises are to be solved individually: **it is your examination!**

The exercises to be done by each of you are parametrised by $(M_1 - M_2, D_1 - D_2, Y_1 - Y_2)$, which are the moth, day and year of your birthday, but if someone is born year 2000, for this course the student is born 1998. Mine is (1-2, 1-5, 6-3). Some one born March 3 1990 has coordinates (0-3, 0-3, 9-0).

To every one: no-one is born a 10th, 20th or 30th, in those cases you are born the 11th, 21st or 29th. In the same way nobody is born 1990, 1980 or 2000 but 1989 resp. 1979 or 2005 ..

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type, you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \bmod 3$$

where $l = 1$ for an exercise type **a)**, $l = -1$ for an exercise type **b)**, and $l = 0$ for type **c)**.

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

Exercise 1 *In applications surfaces are given as level surfaces. One should be able to calculate the characteristic lines of a level surface, then: Let S be a level surface with equation $F(x, y, z) = 0$.*

a) *Show that the asymptotic curves have equations $dxdF_x + dydF_y + dzdF_z = 0$, $F_x dx + F_y dy + F_z dz = 0$.*

b) *Show that the lines of curvature have equations $F_x dx + F_y dy + F_z dz = 0$,*
$$\begin{vmatrix} F_x & dF_x & dx \\ F_y & dF_y & dy \\ F_z & dF_z & dz \end{vmatrix} = 0$$

c) *Show that the geodesics, a geodesic is a curve with geodesic curvature constantly 0, have equations $F_x dx + F_y dy + F_z dz = 0$,* $\begin{vmatrix} F_x & d^2x & dx \\ F_y & d^2y & dy \\ F_z & d^2z & dz \end{vmatrix} = 0$

Exercise 2 *Calculate the I-, II-fundamental forms and the Weingarten map of the following surfaces charts:*

a) *(catenoid)* $\mathbf{x} = (Y_2 cosh(t/Y_2)\cos\theta, Y_2 cosh(t/Y_2)\sin\theta, Y_2 t)$.

b) *(pseudosphere)* $\mathbf{x} = (D_2 \sin t \cos\theta, D_2 \sin t \sin\theta, D_2(lntan(t/2) + \cos t))$.

c) *(helicoid)* $\mathbf{x} = (t\cos\theta, t\sin\theta, f(t) + Y_2\theta)$. *A helicoid satisfying $f(t) \equiv 0$ is called a right helicoid.*

d) *Hyperbolic paraboloid with equation $z = D_2 xy$.*

Exercise 3 *a1 Consider the two families of curves* $\theta = \pm \ln(\tan(t/2)) + C$ *on the pseudosphere* $\mathbf{x} = (\sin t \cos\theta, \sin t \sin\theta, \ln(\tan(t/2)) + \cos t)$. *Show that all segments of curves in a family bounded by two fixed curves in the other family have equal length.*

a2 Show with one example that the mean curvature is not preserved by a local isometry.

a3 Show that a local isometry that preserves the mean curvature preserves the principal curvatures at corresponding points.

b1 Show that if a surface is tangent to a plane along a curve γ*, then all the points in* γ *are parabolic points on the surface.*

b2 Consider the surface given by the equation $f(x - az, y - bz) = 0$*, with* a *and* b *constants. Show that the tangent plane at any point on the surface is parallel to a fixed direction.*

b3 Show that if all the normal lines to the surface are parallel along a curve γ*, then all the points in* γ *are parabolic.*

Exercise 4 *Calculate the -fundamental form for the following charts of surfaces:*

1. $\mathbf{x}(t,\theta) = (cosh(t/)\cos\theta, cosh(t)\sin\theta, sinh(t))$
2. $\mathbf{y}(t,\theta) = (\cos\theta - t\sin(\theta), \sin\theta + t\cos(\theta), t)$
3. $\sigma(x,y) = (x, y, \sqrt{x^2 + y^2 - 1}),\ x^2 + y^2 > 1$

   Show that the three surfaces are (locally) isometric, in fact diffeomorphic and give an equation of the surface.

Exercise 5 *The parametric curves of a chart* $\mathbf{x}(u_1, u_2)$ *form a net of Chebychev if for any quadrilateral formed by parametric lines, opposite sides have equal lenths. The chart is called of Chebychev. Show that a chart is of Chebychev iff the entries* g_{11}, g_{22} *satisfy that* $\partial g_{11}/\partial u_2 = \partial g_{22}/\partial u_1 = 0$.

Exercise 6 *Consider the surface* $\mathbf{x}(u_1, u_2) = (u_1 \sin(\phi)\cos(u_2), u_1 \sin(\phi)\sin(u_2), u_1\cos(\phi))$*, with* ϕ *a parameter.*

a Show that $\mathbf{x}(u_1, u_2)$ *is (part of) a cone of angle* 2ϕ*. Show that given a fixed angle* a*, the curve* $\gamma(u_2) = \mathbf{x}(e^{u_2 \sin(\phi) cotang(a)}, u_2)$ *intersects all the first parametric curves* $u_2 = ct$ *under constant angle. Which angle?*

b Determine the II-fundamental form and Weingarten maps of $\mathbf{x}(u_1, u_2)$*. How are the points in the surface?*

c Consider a surface of revolution $\mathbf{x} = (f(s)\cos(\theta), f(s)\sin(\theta), g(s))$*, where* $\alpha(s)$ *is a unit speed curve on the* x_1x_3*-plane. Show that* $K \equiv 0$ *iff the meridians are straight lines.*

d *Consider a curve* γ *on a chart* $\mathbf{x}$*. Let* $\overline{\mathbf{x}}$ *and* $\overline{\gamma}$ *be the transformate of* $\mathbf{x}$ *and* γ *resp. by the Gauss map. Show that* γ *is a line of curvature iff* γ *and* $\overline{\gamma}$ *have parallel tangent lines at corresponding points.*

Exercise 7 *Let* $\mathbf{x}(u, v) = (u^2 + v^2, u^2 - v^2, uv)$*, with* $|u| + |v| \neq 0$*.*

a) *Determine I- and II-fundamental forms for* $\mathbf{x}(u, v)$*.*

b) *Find the differential for the arc length of the curves* $u = 2$*,* $v = 1$ *and* $v = au$*. Calculate the length of the curve* $v = au$ *between the intersection points with the curves* $u = 2$ *and* $v = 1$*.*

Exercise 8 *a1 Show that* $H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n d\theta$*, where* θ *is the angle in Euler's Th..*

a2 *Let* $X, Y \in T_P S$ *be two orthogonal tangent vectors. Show that* $H = \frac{1}{2}(II(X, X) + II(Y, Y))$*.*

a3 *Let* $P = \mathbf{x}(u_0, v_0)$ *non-umbilic points on* $\mathbf{x}(u, v)$ *with* $H_P = 0$*. Show that* $T_P\mathbf{x}$ *contains two asymptotic directions orthogonal to each other*

b1 *Consider a point* P *and a chart* $\mathbf{x}$ *around* P *on a surface* S*. We say that two directions* X, Y *in* $T_P S$ *are conjugate if* $II_{\mathbf{x}}(X, Y) = 0$*. Their integral curves are called conjugate curves. Determine the conjugate directions and conjugate curves to the parametric curves of the hyperbolic paraboloid with equation* $z = xy$*.*

b2 *Assume that* $K \neq 0$ *on a surface. Show that for any non-zero vector* $X \in T_P S$ *there are exactly two unitary vectors conjugate to* X *in* $T_P S$*.*

b3 *Show that* $H^2 \geq K$*. When is the equality attained?*

c *Determine the elliptic, parabolic and hyperbolic points on a torus* $\mathbf{x}(s, \theta) = ((a + b\cos(s))\cos(\theta), (a + b\cos(s))\sin(\theta), b\sin(s))$*, with* a, b *constants such that* $a + b\cos(s) > 0$*.*

Exercise 9 *Prove that a diffeomorphism between two surfaces that is conformal and equiareal is an isometry.*

Exercise 10 *Determine the lines of curvature on the following surfaces:*
Catenoid: $\mathbf{x} = (acosh(t/a)\cos\theta, acosh(t/a)\sin\theta, t)$*,*

a1 $\mathbf{x} = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$*,*

a2 *Right helicoid:* $\mathbf{x} = (t\cos\theta, t\sin\theta, a\theta)$*,* a *a constant.*

Determine the asymptotic curves on the following surfaces:

b1 *(catenoid)* $\mathbf{x} = (Y_2 cosh(t/Y_2)\cos\theta, Y_2 cosh(t/Y_2)\sin\theta, Y_2 t)$*,*

b2 $\mathbf{x} = (u - \frac{u^3}{3} + D_2 uv^2, v - \frac{v^3}{3} + D_2 vu^2, y_4(u^2 - v^2))$*,* *b3 Right helicoid,* $\mathbf{x} = (t\cos\theta, t\sin\theta, Y_2\theta)$.

Compute the principal directions of the surfaces:

c1 $\mathbf{x}(u,v) = (u+v, u^2+uv, u)$

c2 the surface of Monge with equation $z = f(u)$*, with* $u = \sqrt{x^2-y^2}$*, where it can be defined.*

c3 The surface with equation $z = xy$*, where it can be defined*

Exercise 11 **a)** *Consider the surface given by the equation* $f(x-az, y-bz) = 0$*, with* a *and* b *constants. Show that the tangent plane at any point on the surface is parallel to a fixed direction.*

b) *A simple surface* $\mathbf{r}(s,t)$ *is called a surface of translation if* $\mathbf{r}(s,t) = \mathbf{x}(s) + \mathbf{y}(t)$*. Show that the tangent planes along a parameter curve are parallel to a fixed direction.*

c) *show that the surfaces given by equations* $z = tan(x/y)$ *and* $x^2 - y^2 = a$ *are orthogonal.*

d) *Show that regions on the paraboloids* $z = (x^2+y^2)/2$ *and* $z = xy$ *that project on regions on the plane with equal area have the same area.*

Exercise 12 **a)** *Determine the orthogonal curves to the (straight) parameter lines on a hyperbolic paraboloid* $z = axy$*.*

b) *Calculate the angle of intersection of the (straight) parameter lines at a generic point of a hyperbolic paraboloid* $z = axy$*.*

c) *Determine the orthogonal curves to the family of curves* $t = Ce^\theta$ *on the helicoid* $\mathbf{x} = (t\cos\theta, t\sin\theta, t+\theta)$*.*