# Hand-in Exercises TATA74 Surfaces 2

To every one: no-one is born a 10th, 20th or 30th, in those cases you are born the 11th, 21st or 29th. In the same way nobody is born 1990, 1980 or 2000 but 1989 resp. 1979 or 2005 ..

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 4, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \bmod 3$$

where $l = 1$ for an exercise type **a)**, $l = -1$ for an exercise type **b)**, and $l = 0$ for an exercise of type **c)**.

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

**Exercise 1** **a)** *Show that the differential equations for a geodesic $\gamma(t) = \mathbf{x}(u(t), v(t))$ that is not parametrised by its arc-length are*

$$\left\{ \begin{array}{rcl} \ddot{u} + \sum \Gamma^1_{ij}\dot{u}\dot{v} & = & -\dot{u}\frac{d^2t}{ds^2}(\frac{ds}{dt})^2 \\ \ddot{v} + \sum \Gamma^2_{ij}\dot{u}\dot{v} & = & -\dot{v}\frac{d^2t}{ds^2}(\frac{ds}{dt})^2 \end{array} \right\}.$$

**b)** *Let $X$ be a unit vector in $T_PS$. The torsion of a geodesic through a point $P$ with direction $X$ is called* ***the geodesic torsion*** *$\tau_g$ in the $X$-direction. Show that the geodesic torsion to a curve on a chart $\mathbf{x}(u, v)$ can be calculated $\tau_g = [N, d\mathbf{x}, dN]$ (Observe that $\mathbf{x}(u(s), v(s))$)*

**c)** *Show that a geodesic $\gamma$ on a chart $\mathbf{x}$ is a line of principal curvature iff $\gamma$ is a plane curve.*

**Exercise 2**

1. *Consider the surface $\boldsymbol{x}(u, v) = (v, uv, f(u))$. Determine the function $f(u)$ such that the sume of the radie $(\frac{1}{\varkappa_1}, \frac{1}{\varkappa_2})$ of principal curvatures at any point is constantly $0$.*

2. *Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Show that a tangent vector $v = (dx, dy, dz)$ at $P = (p_1, p_2, p_3)$ is a direction of principal curvature if and only if it satisfies*

$$p_1(b^2 - c^2)dydz + p_2(c^2 - a^2)dxdz + p_3(a^2 - b^2)dxdy = 0$$

**Exercise 3** **a)** *Let $P$ be a point on a surface such that any neighbourhood of $P$ contains two families of geodesics that cut each other under constant angle. Determine $K$ at the point $P$.*

**b)** *Let $S$ be a regular, compact, orientable surface which is not homeomorphic to the sphere. Prove that there are points on $S$ where the Gaussian curvature is positive, negative and zero.*

**c)** *Consider a surface of revolution* $\mathbf{x}$*. Show that $K \equiv 0$ iff the meridians are straight lines.*

**d)** *Through a point $P$ of a surface $M$ pass two orthogonal straight lines. Show that the mean curvature at $P$ is 0 ($H_P = 0$).*

**Exercise 4** **a1)** *Let $S$ be a surface such that the curve $\gamma(t)$ with support $\{(x(t), y(t), z(t)); x = y = z^2\}$ is a geodesic. Determine the tangent plane to $S$ at $P(1, 1, 1)$.*

**a2)** *Consider the chart $\mathbf{x} : \mathcal{U} = \mathbb{R} \times (-\pi/2, \pi/2) \to \mathbb{R}^3$given by $\mathbf{x}(t, \theta) = ((t^2 + 1)\cos(\theta), (t^2 + 1)\sin(\theta), t^3 + t)$. Is any value of $a \in \mathbb{R}$ such that the curve $\gamma(\theta) = ((a^2 + 1)\cos(\theta), (a^2 + 1)\sin(\theta), a^3 + a)$ is a geodesic in $\mathbf{x}(\mathcal{U})$?*

**a3)** *Consider the chart $\mathbf{x}(t, \theta) = ((\cos(t) + 2)\cos(\theta), (\cos(t) + 2)\sin(\theta), t)$. Determine the geodesics that are plane curves contained in the plane $\{z = 0\}$.*

**b)** *Consider a curve $\gamma$ on a chart $\mathbf{x}$. Let $\overline{\mathbf{x}}$ and $\overline{\gamma}$ be the transformated of $\mathbf{x}$ and $\gamma$ resp. by the Gauss map. Show that $\gamma$ is a line of curvature iff $\gamma$ and $\overline{\gamma}$ have parallel tangent lines at corresponding points.*

**c)** *Let $S$ be the graph of a smooth function $F$ (i.e. $S$ has a parametrization $z = F(x, y)$) Determine the Gaussian curvature of $S$ in terms of the derivatives of $F$.*

**Exercise 5** *Calculate the Christoffel symbols the geodesics and lines of principal curvature for:*

**a1)** *A torus* $\mathbf{x}(s, \theta) = ((2 + \cos t)\cos\theta, (2 + \cos t)\sin\theta, \sin t)$

**a2)** *A surface with equation $z = f(x, y)$.*

**a3)** *A surface generate by the binormal lines to a curve $\alpha(s)$*

**b1)** *catenoid,* $\mathbf{x} = (acosh(t/a)\cos\theta, acosh(t/a)\sin\theta, t)$,

**b2)** $\mathbf{x}(u, v) = (u + v, u^2 + uv, u)$,

**b3)** *right helicoid,* $\mathbf{x} = (t\cos\theta, t\sin\theta, a\theta)$, *$a$ a constant.*

**c1)** $\mathbf{x} = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$

**c2)** *Let $\boldsymbol{x}(u, v)$ be a parametrised surface with I fundamental form given by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & \cos^2 u \end{pmatrix}$. Calculate the Christoffel symbols, the Gaussian curvature and the equation of the geodesics.*

**c3)** *Show that if the plane curve $\gamma(s) = \mathbf{x}(u(s), v(s))$ is a geodesic, not a straight line, then $\gamma(s) = \mathbf{x}(u(s), v(s))$ is a line of principal curvature in $\boldsymbol{x}(u, v)$.*

**Exercise 6** *Consider the surfaces $\mathbf{x} = (t\cos(\theta), t\sin(\theta), \log(t))$ and $\mathbf{y} = (t\cos(\theta), t\sin(\theta), \theta)$.*

**a)** *Show that $\mathbf{x} =$ and $\mathbf{y}$ are not locally isometric.*

**b)** *Show that their Gaussian curvature at corresponding points coincide. We see then that the hypotesis of constant Gaussian curvature in Minding Th. cannot be released.*

**Exercise 7** *Let $\boldsymbol{x}(u,v)$ be a parametrised surface with I fundamental form $\begin{pmatrix} 1 & 0 \\ 0 & g(u,v) \end{pmatrix}$. Show that the curves of parameter $v$ determine segments on the $u$-curves of the same length. Calculate the length of any $u$-curve.*

**Exercise 8** **a)** *Let $S$ be the embedded surface given as the as the image of the open unit disc in $\mathbb{R}^2$ by the chart $\mathbf{x}(u,v) = (u, v, \ln(1 - u^2 - v^2))$. Prove that $\int_S KdA = 2\pi$.*
**b)** *The habitants of a planet with constant Gauss curvature $K$ measurer the angles of a geodesic triangle and get the following values: 34, 62 and 83 degrees. If the area of the triangle is 2,81 units, determine the Gauss curvature. Is the planet spherical?*

**Exercise 9** *Suppose we have a Riemannian metric on an open disc $D$ of radius $\delta > 0$ centred at the origin in $\mathbb{R}^2$, possibly with $D$ being all in $\mathbb{R}^2$, given by the metric tensor in polar coordinates $\mathbf{x}(r,\lambda)$, $\begin{pmatrix} 1/h^2(r) & 0 \\ 0 & r^2/h(r)^2 \end{pmatrix}$, where $h(r) > 0$ for all $0 \leq r < \delta$*

**a)** *Prove that*

$$K = hh'' - (h')^2 + \frac{hh'}{r}$$

*Calculate $K$ for $\lambda = (u^2 + v^2 + c^2)^{-2}$*

**b)** *Write down the geodesic equation for this metric and show thatany radial curve, parametrized as to have unit spped, is a geodesic in this metric.*

**Exercise 10** *Consider $S = \{(x,y,z)|x^2 + y^2 + z^2 = 1, 0 < x, 0 < y, 0 < z\}$ and the planes , $\pi_1 = \{y = 0\}$, $\pi_2 = \{z = 0\}$, $\pi_3 = \{y = x\}$ and $\pi_4 = \{\sqrt{2}z = 1\}$. Let $\gamma_i$ be the intersection of $S$ with $\pi_i$, for $1 \leq i \leq 4$. Let $M$ be the region enclosed by the union of the arcs $\gamma_i$, $T_1$ the spherical triangle bounded by $\{\pi_1, \pi_2, \pi_3\}$ and $T_2$ the spherical triangle bounded by $\{\pi_1, \pi_3, \pi_4\}$. Calculate the area of $M$, $T_1$ and $T_2$.*

**Exercise 11**

**a)** *By using Gauss-Bonnet Th. show that two geodesics on a surface of negative curvature cannot meet in two points and enclose a simply connected region. Something what happens for the sphere.*

**b)** *Give the Riemann tensor for the plane (equation $z = 0$) and the sphere $x^2 + y^2 + z^2 = 1$.*

**c)** *Consider Poincaré's Upper Half Plane model for the hyperbolic plane with h I fundamental form $\begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Show that the curves $x = \pm \int \frac{ay}{\sqrt{1-a^2y^2}} du$, with $a = const.$ are geodesics.*

**Exercise 12** *Decide if there exists a surface with chart* $\mathbf{x}(u, v)$ *and associated functions*

**a1)** $g_{11} = 1$, $g_{22} = 1$, $g_{12} = 0$, $L_{11} = 1$, $L_{22} = -1$ *and* $L_{12} = 0$*?*

**a2)** $g_{11} = 1$, $g_{22} = e^u$, $g_{12} = 0$, $L_{11} = e^u$, $L_{22} = 1$ *and* $L_{12} = 0$*?*

**a3)** *Calculate the Riemann tensor for a plane, a circular cylinder and a sphere.*

*Let* $R^h_{ijk} = L_{ik}L^h_j - L_{ij}L^h_k$ *be the coefficients to the Riemann tensor. Show that*

**b)** $R^h_{ijk} = -R^h_{ikj}$ *och så* $R^h_{ijj} = 0$, *för alla* $i$, $j$, $k$ *och* $h$.

**c)** $R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0$, *för alla* $i$, $j$, $k$ *och* $h$.