# Lösningar, TATA77, 2020-10-27

1. $y''(t) - 2y'(t) + 2y(t) = 2e^{2t}, \quad t \geq 0, \quad y(0) = 2, \quad y'(0) = 4.$

Enkelsidig laplacetransform ger ($Y = \mathcal{L}_+ y$):

$$s^2 Y(s) - 2s - 4 - 2(sY(s) - 2) + 2Y(s) = \frac{2}{s-2},$$

$$(s^2 - 2s + 2)Y(s) = \frac{2}{s-2} + 2s = \frac{2s^2 - 4s + 2}{s-2},$$

$$Y(s) = \frac{2s^2 - 4s + 2}{(s-2)(s^2 - 2s + 2)} = \frac{1}{s-2} + \frac{s}{s^2 - 2s + 2} =$$

$$= \frac{1}{s-2} + \frac{(s-1)+1}{(s-1)^2 + 1}, \quad \operatorname{Re} s > 2.$$

Tabell och regeln $e^{ct}u(t) \xrightarrow{\mathcal{L}} \hat{u}(s-c)$ ger $y(t)$.

<u>Svar:</u> a) $\dfrac{2s^2 - 4s + 2}{(s-2)(s^2 - 2s + 2)}, \quad \operatorname{Re} s > 2.$

b) $e^{2t} + e^{t}(\cos t + \sin t), \quad t \geq 0.$

2. $u(n) - \sum_{k=0}^{n} 2(-1)^k u(n-k) = n - 1, \quad n \in \mathbb{N}.$

Enkelsidig z-transform ger ($U = \mathcal{Z}_+ u$):

$$U(z) - \frac{2z}{z+1} U(z) = \frac{z}{(z-1)^2} - \frac{z}{z-1}, \quad \frac{1-z}{z+1} U(z) = \frac{z(2-z)}{(z-1)^2},$$

$$U(z) = \frac{z(z+1)(z-2)}{(z-1)^3} = \frac{z^3 - z^2 - 2z}{(z-1)^3}, \quad |z| > 1.$$

Tabell ger $u(n) = \binom{n+2}{2}\chi(n) - \binom{n+1}{2}\chi(n) - 2\binom{n}{2}\chi(n) =$

$$= \left( \frac{(n+2)(n+1)}{2!} - \frac{(n+1)n}{2!} - \frac{2n(n-1)}{2!} \right) \chi(n)$$

$$= \frac{n^2 + 3n + 2 - n^2 - n - 2n^2 + 2n}{2} \chi(n).$$

<u>Svar:</u> $u(n) = -n^2 + 2n + 1, \quad n \in \mathbb{N}.$

3. a) $\hat{u}(\omega) = \dfrac{i\omega^2}{\omega+3i} = i\omega + 3 - \dfrac{9i}{\omega+3i} = i\omega + 3 - \dfrac{9}{3-i\omega}$.

Svar: $u(t) = \delta'(t) + 3\delta(t) - 9e^{3t}\chi(-t)$.

b) $tu = 2\delta' + 3t + 4$, $\quad tu = t(-\delta'' + 3 + 4\underline{t}^{-1})$,

vilket ger: Svar: $u = -\delta'' + 3 + 4\underline{t}^{-1} + C\delta$, $C \in \mathbb{C}$.

c) $\langle tu, \varphi\rangle = \langle u, t\varphi\rangle = -\int_0^\infty (\ln t)(t\varphi(t))'\,dt =$

$= -\left[(\ln t)t\varphi(t)\right]_0^\infty + \int_0^\infty \frac{1}{t} t\varphi(t)\,dt =$

$= -0 + 0 + \int_0^\infty \varphi(t)\,dt = \langle \chi, \varphi\rangle$, $\quad \varphi \in D(\mathbb{R})$.

Svar: $\chi$.

4. $u(t) = e^t$, $0 \le t < \pi$, $T = \pi \Rightarrow \Omega = 2\pi/T = 2$.

a) $\hat{u}(n) = \frac{1}{\pi}\int_0^\pi e^t e^{-in2t}\,dt = \frac{1}{\pi}\left[\frac{e^{(1-i2n)t}}{1-i2n}\right]_0^\pi = \frac{e^\pi - 1}{\pi(1-i2n)}$, $n \in \mathbb{Z}$.

Svar: $\displaystyle\sum_{n=-\infty}^{\infty} \frac{e^\pi - 1}{\pi(1-i2n)} e^{i2nt}$.

b) $u$ har gen. höger- och vänsterderivata i $t=\pi$, så satsen om punktvis konvergens ger att $u$:s f.s. summa i $t = \pi$ är $\dfrac{u(\pi+)+u(\pi-)}{2} = \dfrac{u(0+)+e^\pi}{2} = \dfrac{1+e^\pi}{2}$.

c) Parsevals formel ger: $\frac{1}{\pi}\int_0^\pi |e^t|^2\,dt = \displaystyle\sum_{n=-\infty}^{\infty}\left|\frac{e^\pi - 1}{\pi(1-i2n)}\right|^2$.

$VL = \frac{1}{\pi}\int_0^\pi e^{2t}\,dt = \frac{1}{\pi}\left[\frac{e^{2t}}{2}\right]_0^\pi = \dfrac{e^{2\pi}-1}{2\pi} = \dfrac{(e^\pi+1)(e^\pi-1)}{2\pi}$,

$HL = \dfrac{(e^\pi - 1)^2}{\pi^2}\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{1+4n^2}$, så:

Svar: $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{4n^2+1} = \frac{\pi}{2}\,\frac{e^\pi+1}{e^\pi-1}$.

5. $y''(t) - 4y'(t) + 3y(t) = 4e^t \chi(2-t)$, $y \in \mathcal{D}'(\mathbb{R})$.

Laplacetransform ger:

$(s^2-4s+3)\hat{y}(s) = \left(-\frac{4e^{-2(s-1)}}{s-1}, \operatorname{Re} s<1\right)_{\mathcal{H}'}$.

$s^2-4s+3 = (s-1)(s-3)$, så

| | | |
|---|---|---|
| $\chi(t)$ | $\xrightarrow{\mathcal{L}}$ | $\frac{1}{s}$, $\operatorname{Re} s>0$ |
| $\chi(-t)$ | | $\frac{1}{-s}$, $\operatorname{Re}(-s)>0$ |
| $\chi(-(t-2))$ | | $-\frac{e^{-2s}}{s}$, $\operatorname{Re} s<0$ |
| $e^t\chi(2-t)$ | | $-\frac{e^{-2(s-1)}}{s-1}$, $\operatorname{Re} s<1$ |

$$\hat{y}(s) = \left(e^{-2s+2}\frac{-4}{(s-1)^2(s-3)}, \operatorname{Re} s<1\right)_{\mathcal{H}'} + 2\pi C\delta_1(s) + 2\pi D\delta_3(s).$$

$$\frac{-4}{(s-1)^2(s-3)} = \frac{2}{(s-1)^2} + \frac{1}{s-1} - \frac{1}{s-3},$$

| | | |
|---|---|---|
| $t\chi(t)$ | $\xrightarrow{\mathcal{L}}$ | $\frac{1}{s^2}$, $\operatorname{Re} s>0$ |
| $-t\chi(-t)$ | | $\frac{1}{(-s)^2}$, $\operatorname{Re}(-s)>0$ |
| $-te^t\chi(-t)$ | | $\frac{1}{(s-1)^2}$, $\operatorname{Re} s<1$ |

så inverstransform (mha regeln $u(t-a) \xrightarrow{\mathcal{L}} e^{-as}\hat{u}(s)$) ger:

$$y(t) = e^2\left(-2te^t\chi(-t) - e^t\chi(-t) + e^{3t}\chi(-t)\right)\Big|_{t\mapsto t-2} + Ce^t + De^{3t}.$$

Svar: $y(t) = \left(e^{3t-4} + (3-2t)e^t\right)\chi(2-t) + Ce^t + De^{3t}$, $C, D \in \mathbb{C}$.

6. $y = Sx = h*x$ ger att $\hat{y}(\omega) = \hat{h}(\omega)\hat{x}(\omega)$.

$x(t) = \begin{cases} 1, & |t|\le 1, \\ 0, & |t|>1, \end{cases}$ så $\hat{x}(\omega) = \frac{2\sin\omega}{\omega}$, enligt tabell.

$$y(t) = (e^{2t}-e^{-2t})e^{-t^2} = e^{-t^2+2t} - e^{-t^2-2t} =$$

$$= e^{-(t-1)^2+1} - e^{-(t+1)^2+1},$$ så tabell ger

$$\hat{y}(\omega) = e\cdot e^{-i\omega}\sqrt{\pi}\,e^{-\omega^2/4} - e\,e^{i\omega}\sqrt{\pi}\,e^{-\omega^2/4}.$$

Alltså har vi $-e\cdot 2i(\sin\omega)\sqrt{\pi}\,e^{-\omega^2/4} = \hat{h}(\omega)\cdot\frac{2\sin\omega}{\omega}$,

så $\hat{h}(\omega) = -i\omega e\sqrt{\pi}\,e^{-\omega^2/4}$, eftersom $h \in L^1(\mathbb{R})$.

Nu fås $h(t) = -\frac{d}{dt}\left(e e^{-t^2}\right) = -e\cdot e^{-t^2}(-2t)$.

Svar: $h(t) = 2te^{1-t^2}$, $t \in \mathbb{R}$.

7. $u(t) = e^{ie^{-t}}, \quad t \in \mathbb{R}.$

$u$ är begränsad, så $u_{D'}$ är tempererad, dvs $0 \in \Sigma_{u_{D'}}$.

Vi har $\left(e^{ie^{-t}} e^{-nt}\right)' = e^{ie^{-t}} ie^{-t}(-1)e^{-nt} + e^{ie^{-t}} e^{-nt}(-n)$,

så $u_{D'} e^{-(n+1)t} = i(u_{D'} e^{-nt})' + in\, u_{D'} e^{-nt}$, vilket ger

att $n \in \Sigma_{u_{D'}} \Rightarrow n+1 \in \Sigma_{u_{D'}}$, så $0, 1, 2, 3, \ldots \in \Sigma_{u_{D'}}$.

Alltså har vi $\sigma^-_{u_{D'}} \leq 0$ och $\sigma^+_{u_{D'}} = \infty$.

Men om $-1 < \sigma < 0$ så är $e^{ie^{-t}} e^{-\sigma t} = \left(1 + \overset{\text{då } t \to \infty}{\mathcal{O}}(e^{-t})\right) e^{-\sigma t} =$

$= e^{-\sigma t} + \underbrace{\mathcal{O}\left(e^{-(1+\sigma)t}\right)}_{\text{begr. då } t \to \infty}$, vilket ger att $u_{D'} e^{-\sigma t}$ inte

är tempererad, så $\sigma^-_{u_{D'}} = 0$.

Alltså ges $u$:s laplacetransform i distributionsmening av en analytisk funktion definierad för $\operatorname{Re} s > 0$.