Lösningar, TATA77, 2021-01-05

1. $y(n+2) - 5y(n+1) + 6y(n) = 2\chi(n-5)$, $n \in \mathbb{N}$, $y(0) = 1$, $y(1) = 4$.

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 1z^2 - 4z - 5(zY(z) - 1z) + 6Y(z) = z^{-5}\frac{2z}{z-1},$$

$$(z^2 - 5z + 6)Y(z) = z^2 - z + z^{-5}\frac{2z}{z-1},$$

$$Y(z) = z\frac{z-1}{(z-3)(z-2)} + z^{-5} \cdot z \frac{2}{(z-3)(z-2)(z-1)} =$$

$$= \frac{2z}{z-3} - \frac{z}{z-2} + z^{-5}\left(\frac{z}{z-3} - \frac{2z}{z-2} + \frac{z}{z-1}\right), \quad |z| > 3.$$

Tabell och räkneregler ger nu

$$y(n) = 2 \cdot 3^n \chi(n) - 2^n \chi(n) + \left[3^n \chi(n) - 2 \cdot 2^n \chi(n) + \chi(n)\right]_{n \mapsto n-5}.$$

Svar: $y(n) = 2 \cdot 3^n - 2^n + (3^{n-5} - 2^{n-4} + 1)\chi(n-5)$, $n \in \mathbb{N}$.

2. $y'(t) + y(t - \frac{\pi}{2}) = \sin^2 t$, $t \in \mathbb{R}$. $T = \pi$, så $\Omega = \frac{2\pi}{T} = 2$.

$$\widehat{VL}(n) = in2\hat{y}(n) + e^{-in2 \cdot \pi/2}\hat{y}(n) = (2in + (-1)^n)\hat{y}(n), \quad n \in \mathbb{Z}.$$

$$HL = \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t = \frac{1}{2} - \frac{1}{4}e^{i2t} - \frac{1}{4}e^{-i2t},$$

så $\widehat{HL}(n) = \begin{cases} 1/2, & n = 0, \\ -1/4, & n = \pm 1, \\ 0, & \text{annars}, \end{cases}$ vilket ger:

$$\hat{y}(n) = \begin{cases} n = 0: & (1/2)/(2i \cdot 0 + 1) = 1/2, \\ n = \pm 1: & -(1/4)/(\pm 2i - 1) = 1/4(1 \mp 2i) = \dfrac{1 \pm 2i}{20}, \\ n \neq 0, \pm 1: & 0/(2in + (-1)^n) = 0. \end{cases}$$

Vi får $y(t) = \sum_{n=-\infty}^{\infty} \hat{y}(n) e^{in2t} = \frac{1}{2} + \frac{1+2i}{20}e^{i2t} + \frac{1-2i}{20}e^{-i2t}$.

Förenkling (med Eulers formler) ger:

Svar: $y(t) = \frac{1}{2} + \frac{1}{10}(\cos 2t - 2\sin 2t)$, $t \in \mathbb{R}$.

3. a) $\langle e^{2t}\delta'(t/3), \varphi(t)\rangle = \langle \delta'(t/3), e^{2t}\varphi(t)\rangle =$

$$= \frac{1}{1/3}\langle \delta'(t), e^{2\cdot 3t}\varphi(3t)\rangle = -3\langle \delta(t), e^{6t}\varphi'(3t)\cdot 3 + 6e^{6t}\varphi(3t)\rangle =$$

$$= -9\varphi'(0) - 18\varphi(0) = \langle 9\delta'(t) - 18\delta(t), \varphi(t)\rangle, \quad \varphi \in D(R).$$

Svar: $9\delta' - 18\delta$.

b) $tu' + u = \delta', \quad (tu)' = \delta', \quad tu = \delta + C = t(-\delta' + C\underline{t}^{-1}),$

så: Svar: $u = -\delta' + C\underline{t}^{-1} + D\delta, \quad C, D \in \mathbb{C}$.

4. $y''(t) - y'(t) - 2y(t) = 4 + 12e^{3t}\chi(-t), \quad y \in S'$.

Fouriertransform ger: $((i\omega)^2 - i\omega - 2)\hat{y}(\omega) = 4\cdot 2\pi\delta(\omega) + \dfrac{12}{3-i\omega}$,

$$\hat{y}(\omega) = \frac{4}{(i\omega+1)(i\omega-2)}\cdot 2\pi\delta(\omega) - \frac{12}{(1+i\omega)(2-i\omega)(3-i\omega)} =$$

$$= -2\cdot 2\pi\delta(\omega) - \frac{1}{1+i\omega} - \frac{4}{2-i\omega} + \frac{3}{3-i\omega}, \quad \omega \in \mathbb{R}.$$

Inverstransform av detta blir:

$$-2 - e^{-t}\chi(t) - 4e^{2t}\chi(-t) + 3e^{3t}\chi(-t),$$

vilket är ett helt ok svar, men samma distribution i S' fås också av följande kontinuerliga funktion:

Svar: $y(t) = -2 - \begin{cases} e^{-t}, & t \geq 0 \\ 4e^{2t} - 3e^{3t}, & t \leq 0 \end{cases}$.

5. $u(t) = e^{-e^{-t}}, \quad t \in \mathbb{R}.$

$u(t) \to 1$ då $t \to \infty$, och $u(t) \to 0$ mycket snabbt då $t \to -\infty$,

så $\hat{u}(s) = \int_{-\infty}^{\infty} e^{-e^{-t}} e^{-st}\,dt$, $\operatorname{Re} s > 0$.

$u'(t) = e^{-e^{-t}} \cdot (-e^{-t})(-1) = e^{-t} u(t)$, så $s\hat{u}(s) = \hat{u}(s+1)$, $\operatorname{Re} s > 0$.

Detta ger $\hat{u}(s) = \dfrac{\hat{u}(s+1)}{s} = \dfrac{\hat{u}(1) + \hat{u}'(1)s + \mathcal{O}(s^2)}{s}$.

$$\hat{u}(1) = \int_{-\infty}^{\infty} e^{-e^{-t}} e^{-t}\,dt = \left[e^{-e^{-t}}\right]_{-\infty}^{\infty} = 1 - 0 = 1.$$

$$\hat{u}'(s) = \int_{-\infty}^{\infty} \frac{d}{ds}\left(e^{-e^{-t}} e^{-st}\right) dt = \int_{-\infty}^{\infty} e^{-e^{-t}} e^{-st}(-t)\,dt,$$

så $\hat{u}'(1) = \int_{-\infty}^{\infty} e^{-e^{-t}} e^{-t}(-t)\,dt = \Big/ \begin{matrix} r = e^{-t} \\ dr = -e^{-t}\,dt \end{matrix} \Big/$

$$= \int_0^{\infty} e^{-r} \ln r\,dr = -\gamma. \quad (\gamma = \text{Eulers konstant, se 10.6.})$$

Alltså har vi $\hat{u}(s) - \frac{1}{s} = -\gamma + \mathcal{O}(s) \to -\gamma$ då $s \to 0+$.

Svar: $c = 1$, och gränsvärdet är $-\gamma$.