Lösningar, TATA77, 2022-01-05

1. $y(n+2) + y(n+1) + y(n) = 7 \cdot 2^n, \quad n \in \mathbb{N}, \quad y(0) = 3, \; y(1) = 1.$

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 3z^2 - z + zY(z) - 3z + Y(z) = \frac{7z}{z-2},$$

$$(z^2+z+1)Y(z) = \frac{7z}{z-2} + 3z^2 + 4z = \frac{3z^3 - 2z^2 - z}{z-2},$$

$$Y(z) = z\,\frac{3z^2-2z-1}{(z-2)(z^2+z+1)} = z\left(\frac{1}{z-2} + \frac{2z+1}{z^2+z+1}\right) =$$

$$= \frac{z}{z-2} + \frac{2z^2+z}{z^2+z+1} = \frac{z}{z-2} + \frac{2\left(z^2 - \left(\cos\frac{2\pi}{3}\right)z\right)}{z^2 - 2\left(\cos\frac{2\pi}{3}\right)z + 1}, \quad |z| > 2.$$

Tabell ger:

<u>Svar:</u> $y(n) = 2^n + 2\cos\frac{2\pi n}{3}, \quad n \in \mathbb{N}.$

2. $u(t) + \int_{-\infty}^{0} 2e^r u(t-r)\,dr = e^{2t}\chi(-t), \quad t \in \mathbb{R}.$

Integralen $= (f*u)(t)$, där $f(t) = 2e^t\chi(-t)$, $t \in \mathbb{R}$, så

fouriertransform ger: $\quad \hat{u}(\omega) + \frac{2}{1-i\omega}\hat{u}(\omega) = \frac{1}{2-i\omega}, \quad \omega \in \mathbb{R}.$

Detta ger $\hat{u}(\omega) = \frac{1-i\omega}{(3-i\omega)(2-i\omega)} = \frac{2}{3-i\omega} + \frac{-1}{2-i\omega}$, så:

<u>Svar:</u> $u(t) = (2e^{3t} - e^{2t})\chi(-t), \quad t \in \mathbb{R}.$

3. $u(t) = t^2, \quad -\pi \le t < \pi, \quad T = 2\pi$ så $\Omega = 1$.

$$\hat{u}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-int}\,dt \overset{n \neq 0}{=} \frac{1}{2\pi}\left[t^2\frac{e^{-int}}{-in} - 2t\frac{e^{-int}}{(-in)^2} + 2\frac{e^{-int}}{(-in)^3}\right]_{-\pi}^{\pi} = \frac{2(-1)^n}{n^2}, \quad n \neq 0,$$

och $\hat{u}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{\pi^2}{3}.$

<u>Delsvar:</u> Fourierserien är $\frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2(-1)^n}{n^2} e^{int}.$

Parsevals formel ger:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |t^2|^2\,dt = \frac{\pi^4}{9} + \sum_{n\neq 0}\frac{4}{n^4}, \quad \text{så} \quad \frac{\pi^4}{5} = \frac{\pi^4}{9} + \sum_{n=1}^{\infty}\frac{8}{n^4}, \quad \text{så:}$$

<u>Delsvar:</u> $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$

4. $y'' - y' - 2y = x' - x$ då $y = Sx$ ger att $h'' - h' - 2h = \delta' - \delta$.

Laplacetransform ger att $(s^2 - s - 2)\hat{h}(s) = s - 1$, så

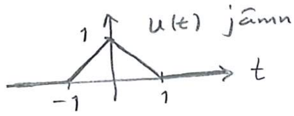
$$\hat{h}(s) = \left(\frac{s-1}{(s-2)(s+1)}, \operatorname{Re} s > 2\right)_{\mathcal{H}'} + 2\pi C \delta_2(s) + 2\pi D \delta_{-1}(s).$$

$$\frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1} \quad (s \neq 2, -1), \text{ vilket ger:}$$

$$h(t) = \tfrac{1}{3} e^{2t} \chi(t) + \tfrac{2}{3} e^{-t} \chi(t) + C e^{2t} + D e^{-t}, \quad C, D \in \mathbb{C}.$$

h begränsad $\Rightarrow$ $C = -\frac{1}{3}$, $D = 0$ och vi får:

<u>Svar:</u> $h(t) = -\frac{1}{3} e^{2t} \chi(-t) + \frac{2}{3} e^{-t} \chi(t)$, $t \in \mathbb{R}$.



5. $u(t) = 1 - |t|$ då $|t| \leq 1$, $u(t) = 0$ då $|t| > 1$.

$$\hat{u}(\omega) = \int_{-1}^{1} u(t) e^{-i\omega t}\, dt = \int_{-1}^{1} u(t)(\cos \omega t - i \sin \omega t)\, dt =$$

$$= 2\int_0^1 (1-t)\cos \omega t\, dt = 2\left[(1-t)\frac{\sin \omega t}{\omega} - (-1)\frac{-\cos \omega t}{\omega^2}\right]_0^1 =$$

$$= \frac{2(1-\cos\omega)}{\omega^2}, \text{ vilket alltså är } u\text{:s fouriertransform (då } \omega \neq 0).$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos\omega) e^{-i\omega} \sin\omega}{\omega^3}\, d\omega = \int_{-\infty}^{\infty} \hat{u}(\omega) \overline{\frac{e^{i\omega}\sin\omega}{2\omega}}\, d\omega = /\text{Parseval}/$$

$$= 2\pi \int_{-\infty}^{\infty} u(t)\overline{v(t)}\, dt, \text{ där } \hat{v}(\omega) = \frac{e^{i\omega}\sin\omega}{2\omega} \quad (\omega \neq 0).$$

Tabell och räkneregler ger $v(t) = \begin{cases} 1/4, & -2 \leq t \leq 0, \\ 0, & \text{annars,} \end{cases}$ så

$$2\pi \int_{-\infty}^{\infty} u(t)\overline{v(t)}\, dt = 2\pi \int_{-1}^{0} (1+t)\tfrac{1}{4}\, dt = \frac{\pi}{4}.$$

Alltså fås $\displaystyle\int_{-\infty}^{\infty} \frac{(1-\cos\omega)e^{-i\omega}\sin\omega}{\omega^3}\, d\omega = \frac{\pi}{4}$.

6. Sätt $u(t) = \sum_{n=-\infty}^{\infty} \frac{e^{int}}{n^2+2i}$. $u \in D'_{2\pi}$ och $u''(t) = \sum_{n=-\infty}^{\infty} \frac{-n^2 e^{int}}{n^2+2i}$,

så $-u'' + 2iu = \sum_{n=-\infty}^{\infty} e^{int} = \sum_{n=-\infty}^{\infty} 2\pi \delta_{2\pi n}$.

Alltså är $u'' - 2iu = -2\pi\delta$ på $]-2\pi, 2\pi[$.

Kar.ekv. $r^2 - 2i = 0$ ger $r = \pm(1+i)$, så $u_h = Ae^{(1+i)t} + Be^{-(1+i)t}$.

$u_f = Ce^{(1+i)t} + De^{-(1+i)t}$ och $u_f(0) = 0$, $u_f'(0) = 1$ ger $C = -D = \frac{1}{2(1+i)}$,

så $f = u_f \chi = \frac{1}{2(1+i)}(e^{(1+i)t} - e^{-(1+i)t})\chi$ uppfyller $f'' - 2if = \delta$.

Detta ger att $u = -\frac{\pi}{1+i}(e^{(1+i)t} - e^{-(1+i)t})\chi(t) + u_h(t)$ på $]-2\pi, 2\pi[$.

Eftersom $u(t) = u(t-2\pi)$ för $0 < t < 2\pi$ fås att

$$-\frac{\pi}{1+i}(e^{(1+i)t} - e^{-(1+i)t}) + u_h(t) = u_h(t-2\pi), \text{ vilket ger att} \ldots$$

$$A = \frac{\pi}{(1+i)(1-e^{-2\pi})}, \quad B = \frac{\pi}{(1+i)(e^{2\pi}-1)}.$$

Nu fås $u(0) = A + B = \frac{\pi}{1+i}\left(\frac{e^{2\pi}}{e^{2\pi}-1} + \frac{1}{e^{2\pi}-1}\right)$.

Svar: $\frac{\pi(e^{2\pi}+1)}{(1+i)(e^{2\pi}-1)}$.

7. $u \in L^1_T$, och det finns ett $v \in L^2_T$ s.a. $(u_{D'})' = v_{D'}$.

Cauchy-Schwarz

$$\sum_{n=-\infty}^{\infty} |\hat{u}(n)| = |\hat{u}(0)| + \sum_{n\neq 0} |\hat{u}(n)| = |\hat{u}(0)| + \sum_{n \neq 0} \left|\frac{1}{in\Omega} in\Omega\hat{u}(n)\right| \leq$$

$$\leq |\hat{u}(0)| + \left(\sum_{n\neq 0}\left|\frac{1}{in\Omega}\right|^2\right)^{1/2}\left(\sum_{n\neq 0}|in\Omega\hat{u}(n)|^2\right)^{1/2} = \Big/\ \widehat{(u_{D'})'}(n) = in\Omega\hat{u}(n)\ \Big/$$

$$= |\hat{u}(0)| + \frac{1}{\Omega}\left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=-\infty}^{\infty}|\hat{v}(n)|^2\right)^{1/2} = /\ \text{Parseval}\ /$$

$$= |\hat{u}(0)| + \frac{1}{\Omega}\left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\|v\|_{2,T} < \infty, \text{ så } u\text{:s fourierserie}$$

är absolutkonvergent.