Lösningar, TATA77, 2022-08-24

1. $y''(t)+3y'(t)+2y(t)=3e^{-t},\ t\geq 0,\ y(0)=3,\ y'(0)=-2.$

Enkelsidig laplacetransform ger $(Y=\mathcal{L}_+y)$:

$$s^2Y(s)-3s+2+3(sY(s)-3)+2Y(s)=\frac{3}{s+1},$$

$$(s^2+3s+2)Y(s)=\frac{3}{s+1}+3s+7=\frac{3s^2+10s+10}{s+1},$$

$$Y(s)=\frac{3s^2+10s+10}{(s+1)^2(s+2)}=\frac{3}{(s+1)^2}+\frac{1}{s+1}+\frac{2}{s+2},\quad \operatorname{Re} s>-1.$$

Tabell ger $y(t)$. Svar: a) $\dfrac{3s^2+10s+10}{(s+1)^2(s+2)},\ \operatorname{Re} s>-1.$

b) $(3t+1)e^{-t}+2e^{-2t},\ t\geq 0.$

2. $y'(t)+y(t)=\delta(t)+3e^{-2|t|}\operatorname{sgn} t.$

Fouriertransform ger: $i\omega\hat{y}(\omega)+\hat{y}(\omega)=1+\dfrac{-6i\omega}{4+\omega^2},$

$$\hat{y}(\omega)=\frac{4+\omega^2-6i\omega}{(1+i\omega)(4+\omega^2)}=\frac{3}{1+i\omega}+\frac{2i\omega-8}{4+\omega^2},\quad \omega\in\mathbb{R}.$$

Tabell ger: Svar: $y(t)=3e^{-t}\chi(t)-e^{-2|t|}(\operatorname{sgn} t+2).$

3. a) $u(t)=\operatorname{at}(1/t),\ t\neq 0,\ u(0)=0.$

$$u'(t)=\frac{1}{1+(1/t)^2}\cdot\left(-\frac{1}{t^2}\right)+(u(0+)-u(0-))\delta_0(t)\quad \text{(sats 2.5)},$$

så: Svar: $u'(t)=-\dfrac{1}{t^2+1}+\pi\delta(t).$

b) $\langle e^{2t}\delta'',\varphi\rangle=\langle\delta'',e^{2t}\varphi\rangle=(-1)^2\langle\delta,(e^{2t}\varphi)''\rangle=$

$=\langle\delta,e^{2t}\varphi''+4e^{2t}\varphi'+4e^{2t}\varphi\rangle=\varphi''(0)+4\varphi'(0)+4\varphi(0)=$

$=\langle\delta''-4\delta'+4\delta,\varphi\rangle,\ \varphi\in\mathcal{D}(\mathbb{R}).$ Svar: $\delta''-4\delta'+4\delta.$

c) $u'=\operatorname{sgn} t$, så $u'=1$ på $]0,\infty[$ och $u'=-1$ på $]-\infty,0[$.

Detta ger $u=\begin{cases}t+C, & t>0\\ -t+D, & t<0\end{cases}$ och vi får $u'=\begin{Bmatrix}1, & t>0\\ -1, & t<0\end{Bmatrix}+(C-D)\delta.$

Alltså måste $C-D=0$. Svar: $u=|t|+C,\ C\in\mathbb{C}.$

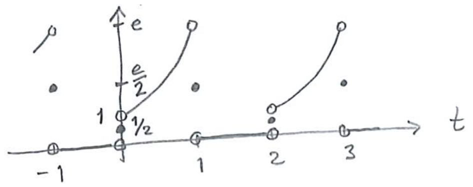
4. $u(t)=e^t,\ 0\le t<1,\ u(t)=0,\ 1\le t<2,\quad T=2 \quad\Rightarrow\quad \Omega=\frac{2\pi}{T}=\pi.$

$$\hat u(n)=\frac{1}{2}\int_0^2 u(t)e^{-in\pi t}\,dt=\frac{1}{2}\int_0^1 e^t e^{-in\pi t}\,dt=\frac{1}{2}\left[\frac{e^{(1-in\pi)t}}{1-in\pi}\right]_0^1=$$

$$=\frac{e^{1-in\pi}-1}{2(1-in\pi)}=\frac{(-1)^n e-1}{2(1-in\pi)},\quad n\in\mathbb{Z}.$$

Delsvar: $\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n e-1}{2(1-in\pi)}\,e^{in\pi t}.$

Fourierseriens summa, enl. satsen om punktvis konvergens:



5. $\displaystyle u(n)+\sum_{k=n}^{\infty}2^{n-k}u(k)=3\chi(n),\quad n\in\mathbb{Z}.$

Vi har $3\chi(n)\xrightarrow{\mathcal{Z}}\frac{3z}{z-1},\ |z|>1.\quad \displaystyle\sum_{k=n}^{\infty}2^{n-k}u(k)=\sum_{k=-\infty}^{\infty}2^{n-k}\chi(k-n)u(k)=$

$=(f*u)(n)$, där $f(n)=2^n\chi(-n)$. Vi har $2^{-n}\chi(n)\xrightarrow{\mathcal{Z}}\frac{z}{z-1/2},\ |z|>\frac{1}{2}$,

så $\hat f(z)=\dfrac{1/z}{1/z-1/2}=\dfrac{2}{2-z},\quad 0<|z|<2.$

$\mathcal{Z}(VL)=\mathcal{Z}(HL)$: $\quad\hat u(z)+\dfrac{2}{2-z}\hat u(z)=\dfrac{3z}{z-1},\quad |z|\in R_u\cap\,]1,2[\,$, så

$$\hat u(z)=\frac{3z(z-2)}{(z-1)(z-4)}=z\left(\frac{1}{z-1}+\frac{2}{z-4}\right)=\frac{z}{z-1}+\frac{2z}{z-4},\quad |z|\in R_u,$$

och vi måste ha $R_u=\,]1,4[$ för att få $R_u\cap\,]1,2[\,\neq\emptyset$.

Vi har: $\quad 4^{-n}\chi(n)\xrightarrow{\mathcal{Z}}\dfrac{z}{z-1/4},\quad |z|>1/4$

$\quad 4^n\chi(-n)\qquad \dfrac{1/z}{1/z-1/4}=\dfrac{4}{4-z}=-\dfrac{4}{z}\cdot\dfrac{z}{z-4},\quad 0<|z|<4$

$\quad 4^{n+1}\chi(-(n+1))\qquad -4\dfrac{z}{z-4},\quad 0<|z|<4$, så:

Svar: $u(n)=\chi(n)-2\cdot 4^n\chi(-n-1),\quad n\in\mathbb{Z}.$

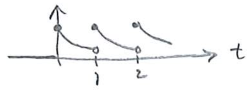
6. $\hat{u}(s) = \dfrac{1-e^{-1-s}}{(s+1)(1-e^{-s})}$, $\operatorname{Re} s > 0$.

$|e^{-s}| < 1$ så $\hat{u}(s) = \dfrac{1-e^{-1-s}}{s+1}\left(1 + e^{-s} + e^{-2s} + \dots\right)$.

Med $\hat{v}(s) = \dfrac{1-e^{-1-s}}{s+1}$, $\operatorname{Re} s > 0$, fås mha tabell att

$$v(t) = e^{-t}\chi(t) - e^{-1}e^{-(t-1)}\chi(t-1) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ 0, & \text{annars}. \end{cases}$$

Så $u(t) = v(t) + v(t-1) + v(t-2) + \dots$:



Svar: $u(t) = \begin{cases} 0, & t < 0, \\ e^{-(t-n)}, & n \le t < n+1, \ n \in \mathbb{N}. \end{cases}$

7. $u(t) = t^2$ då $0 \le t < 1$ och u 1-periodisk.

$$(\mathcal{L}_+u)(s) = \int_0^\infty u(t)e^{-st}\,dt = \sum_{n=0}^{\infty} \int_n^{n+1} u(t)e^{-st}\,dt =$$

$$= \sum_{n=0}^{\infty} \int_0^1 u(t+n)e^{-s(t+n)}\,dt = \sum_{n=0}^{\infty} e^{-sn}\int_0^1 u(t)e^{-st}\,dt =$$

$$= \frac{1}{1-e^{-s}} \int_0^1 t^2 e^{-st}\,dt, \quad \operatorname{Re} s > 0.$$

Så
$$(\mathcal{L}_+u)(2+2\pi i n) = \frac{1}{1-e^{-(2+2\pi i n)}} \int_0^1 t^2 e^{-(2+2\pi i n)t}\,dt =$$

$$= \frac{1}{1-e^{-2}} \int_0^1 t^2 e^{-2t}\, e^{-in2\pi t}\,dt = \frac{e^2}{e^2-1}(\mathcal{F}_T v)(n), \quad n \in \mathbb{Z},$$

där $v(t) = t^2e^{-2t}$, $0 \le t < 1$, och v är 1-periodisk.

Satsen om punktvis konvergens ger att

$$\lim_{N\to\infty} \sum_{n=-N}^{N} (\mathcal{L}_+u)(2+2\pi i n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{e^2}{e^2-1}(\mathcal{F}_T v)(n) =$$

$$= \frac{e^2}{e^2-1}\,\frac{v(0+)+v(0-)}{2} = \frac{e^2}{e^2-1}\cdot\frac{0+e^{-2}}{2} = \underline{\underline{\frac{1}{2(e^2-1)}}}.$$