Lösningar, TATA77, 2022-10-26

1. $y(n+2) - 3y(n+1) + 2y(n) = 4\cdot 3^n$, $n \in \mathbb{N}$, $y(0) = 2$, $y(1) = 5$.

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 2z^2 - 5z - 3(zY(z) - 2z) + 2Y(z) = \frac{4z}{z-3},$$

$$(z^2 - 3z + 2)Y(z) = \frac{4z}{z-3} + 2z^2 - z = \frac{2z^3 - 7z^2 + 7z}{z-3},$$

$$Y(z) = z\,\frac{2z^2 - 7z + 7}{(z-3)(z-2)(z-1)} = z\left(\frac{2}{z-3} + \frac{-1}{z-2} + \frac{1}{z-1}\right) =$$

$$= \frac{2z}{z-3} - \frac{z}{z-2} + \frac{z}{z-1}, \quad |z| > 3. \quad \text{Tabell ger:}$$

<u>Svar:</u> $y(n) = 2\cdot 3^n - 2^n + 1$, $n \in \mathbb{N}$.

2. $y''(t) - 2y'(t) + y(t) = \delta(t+1)$, $y \in \mathcal{D}'(\mathbb{R})$.

Laplacetransform ger: $s^2\hat{y}(s) - 2s\hat{y}(s) + \hat{y}(s) = e^s \cdot 1$, $(s-1)^2\hat{y}(s) = e^s$,

$$\hat{y}(s) = \left(\frac{e^s}{(s-1)^2},\ \operatorname{Re} s > 1\right)_{\mathcal{H}'} + A\delta(s-1) + B\delta'(s-1).$$

$$te^t \chi(t) \xrightarrow{\mathcal{L}} \frac{1}{(s-1)^2},\ \operatorname{Re} s > 1, \quad \text{så} \quad (t+1)e^{t+1}\chi(t+1) \longrightarrow \frac{e^s}{(s-1)^2},\ \operatorname{Re} s > 1.$$

$$1 \xrightarrow{\mathcal{L}} 2\pi\delta(s), \quad \text{så} \quad e^t \to 2\pi\delta(s-1), \quad \text{och} \quad te^t \to -2\pi\delta'(s-1).$$

Detta ger: <u>Svar:</u> $y(t) = (t+1)e^{t+1}\chi(t+1) + Ce^t + Dte^t$, $C, D \in \mathbb{C}$.

3. $u(t) = \cos t$ då $|t| \le \pi/2$, $u(t) = 0$ då $|t| > \pi/2$.

$$\hat{u}(\omega) = \int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-i\omega t}\,dt = \left[\sin t \cdot e^{-i\omega t}\right]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \sin t \cdot e^{-i\omega t}(-i\omega)\,dt =$$

$$= e^{-i\omega\pi/2} - (-1)e^{i\omega\pi/2} + i\omega\left(\left[-\cos t \cdot e^{-i\omega t}\right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-i\omega t}(-i\omega)\,dt\right) =$$

$$= 2\cos(\pi\omega/2) + 0 + \omega^2\hat{u}(\omega), \quad \text{så:}$$

$$\underline{\underline{\hat{u}(\omega) = \frac{2\cos(\pi\omega/2)}{1-\omega^2}}} \quad (\text{för } \omega \neq \pm 1, \text{ men } u \in L^1(\mathbb{R}) \text{ så } \hat{u} \text{ kont.}).$$

$$\int_{-\infty}^{\infty} \frac{\cos(\pi\omega/2)}{1-\omega^2}\,d\omega = \frac{2\pi}{2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\omega)e^{i\omega\cdot 0}\,d\omega \overset{\text{Inversionsformeln}}{=} \pi\cdot u(0) = \underline{\underline{\pi}}.$$

$$\int_{-\infty}^{\infty} \frac{\cos(\pi\omega/2)\cos\omega}{1-\omega^2}\,d\omega = \frac{2\pi}{2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(\omega)\tfrac{1}{2}(e^{i\omega} + e^{-i\omega})\,d\omega = \pi\tfrac{1}{2}(u(1) + u(-1)) =$$

$$= \frac{\pi}{2}\cdot 2\cos 1 = \underline{\underline{\pi\cos 1}}.$$

4. $u(t) + \int_0^t (e^{3r} - e^r)\, u(t-r)\, dr = e^{2t}, \quad t \geq 0.$

Enkelsidig laplacetransform ger $(U = \mathcal{L}_+ u)$:

$$U(s) + \left(\frac{1}{s-3} - \frac{1}{s-1}\right) U(s) = \frac{1}{s-2}, \quad \frac{s^2-4s+3+s-1-(s-3)}{s^2-4s+3}\, U(s) = \frac{1}{s-2},$$

$$U(s) = \frac{s^2-4s+3}{(s^2-4s+5)(s-2)} = \frac{2s-4}{s^2-4s+5} + \frac{-1}{s-2} = \frac{2(s-2)}{(s-2)^2+1} - \frac{1}{s-2}, \quad \operatorname{Re} s > 2.$$

Tabell och regeln $e^{ct} u(t) \xrightarrow{\mathcal{L}} \hat{u}(s-c)$ ger att detta är laplacetr. av

$2e^{2t}(\cos t)\chi(t) - e^{2t}\chi(t)$. <u>Svar:</u> $u(t) = e^{2t}(2\cos t - 1), \quad t \geq 0.$

5. $tu' + 3u = \delta''' + 6\delta + 6, \quad u \in \mathcal{D}'(\mathbb{R}).$

$\Rightarrow \; t^3u' + 3t^2u = t^2(\delta''' + 6\delta + 6)$ (Ej $\Leftrightarrow$, så lösningar måste prövas.)

$\Leftrightarrow \; (t^3u)' = (-3)(-2)\delta' + 0 + 6t^2, \quad t^3u = 6\delta + 2t^3 + A,$

$u = -\delta''' + 2 + A\underline{t}^{-3} + B\delta'' + C\delta' + D\delta.$

Prövning: $u' = -\delta'''' + 0 - 3A\underline{t}^{-4} + B\delta''' + C\delta'' + D\delta',$

$tu' = 4\delta''' - 3A\underline{t}^{-3} - 3B\delta'' - 2C\delta' - D\delta,$

så $tu' + 3u = \delta''' + 6 + C\delta' + 2D\delta$, vilket ger: $C = 0$, $D = 3$.

<u>Svar:</u> $u = -\delta''' + 3\delta + 2 + A\underline{t}^{-3} + B\delta'', \quad A, B \in \mathbb{C}.$

6. Sätt $\varphi(t) = e^{-\pi t^2 \varepsilon}, \quad t \in \mathbb{R}$, där $\varepsilon > 0$. Vi har $(\varphi \in \mathcal{S})$:

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 \varepsilon} = \left\langle \sum_{n=-\infty}^{\infty} \delta(t-n), \varphi(t) \right\rangle = \text{/ Fourierserien för ett pulståg /}$$

$$= \left\langle \sum_{n=-\infty}^{\infty} e^{in2\pi t}, \varphi(t) \right\rangle = \sum_{n=-\infty}^{\infty} \hat{\varphi}(-n2\pi) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{\pi}{\pi\varepsilon}}\, e^{-(-n2\pi)^2/4\pi\varepsilon} =$$

$$= \frac{1}{\sqrt{\varepsilon}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\varepsilon} = \frac{1}{\sqrt{\varepsilon}} + \frac{2}{\sqrt{\varepsilon}} \sum_{n=1}^{\infty} e^{-\pi n^2/\varepsilon}.$$

Så $$\left| \frac{1}{\sqrt{\varepsilon}} - \sum_{n=-\infty}^{\infty} e^{-\pi n^2 \varepsilon} \right| = \frac{2}{\sqrt{\varepsilon}} \sum_{n=1}^{\infty} e^{-\pi n^2/\varepsilon} \leq \frac{2}{\sqrt{\varepsilon}} \sum_{n=1}^{\infty} e^{-\pi n \varepsilon} =$$

$$= \frac{2}{\sqrt{\varepsilon}} \cdot \frac{e^{-\pi/\varepsilon}}{1 - e^{-\pi/\varepsilon}} \longrightarrow 0 \text{ då } \varepsilon \to 0+ \quad \text{(standard).}$$

7. Låt $u(t) = \frac{1}{\pi + t}$ då $0 \le t < 2\pi$ och låt u ha period 2π.

Då är $u(t) = c(t) + s(t)$, där $c(t) = \frac{u(t) + u(2\pi - t)}{2}$ är jämn kring π, och $s(t) = \frac{u(t) - u(2\pi - t)}{2}$ är udda kring π.

För u:s fourierserie $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ gäller (då $t \neq 2\pi m$) att $c(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$ och $s(t) = \sum_{n=1}^{\infty} b_n \sin nt$, eftersom $\cos nt$ ($\sin nt$) är jämn (udda) kring π.

Integralen minimeras då b_n väljs som u:s fourierkoefficienter (ty $\cos mt \perp \sin nt$ i $L^2_{2\pi}$), och det minsta värdet blir:

$$\int_0^{2\pi} |c(t)|^2 = \int_0^{2\pi} \left| \frac{1}{2} \left(\frac{1}{\pi + t} + \frac{1}{3\pi - t} \right) \right|^2 dt =$$

$$= \frac{1}{4} \int_0^{2\pi} \left(\frac{1}{(\pi + t)^2} + \frac{1}{(3\pi - t)^2} + \frac{1/2\pi}{\pi + t} + \frac{1/2\pi}{3\pi - t} \right) = / \text{symm. kring } \pi /$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{1}{(\pi + t)^2} + \frac{1/2\pi}{\pi + t} \right) dt = \frac{1}{2} \left[-\frac{1}{\pi + t} + \frac{1}{2\pi} \ln(\pi + t) \right]_0^{2\pi} =$$

$$= \frac{1}{2} \left(-\frac{1}{3\pi} + \frac{1}{2\pi} \ln 3\pi + \frac{1}{\pi} - \frac{1}{2\pi} \ln \pi \right) = \underline{\underline{\frac{4 + 3\ln 3}{12\pi}}}.$$