Lösningar, TATA77, 2023-10-25

1. $y''(t) + 2y'(t) + 5y(t) = 5e^{-2t}, \quad t \geq 0, \quad y(0) = 2, \quad y'(0) = -3.$

Enkelsidig laplacetransform ger ($Y = \mathcal{L}_+ y$):

$$s^2 Y(s) - 2s - (-3) + 2(sY(s) - 2) + 5Y(s) = \frac{5}{s+2},$$

$$(s^2 + 2s + 5)Y(s) = \frac{5}{s+2} + 2s + 1 = \frac{2s^2 + 5s + 7}{s+2},$$

$$Y(s) = \frac{2s^2 + 5s + 7}{(s+2)(s^2 + 2s + 5)} = \frac{1}{s+2} + \frac{s+1}{s^2 + 2s + 5} = \frac{1}{s+2} + \frac{s+1}{(s+1)^2 + 4}, \quad \text{Re } s > -1.$$

Inverstransform med tabell och räkneregler ger $y(t)$.

Svar: a) $\dfrac{2s^2 + 5s + 7}{(s+2)(s^2 + 2s + 5)}, \quad \text{Re } s > -1.$

b) $e^{-2t} + e^{-t} \cos 2t, \quad t \geq 0.$

2. $\sum_{k=0}^{n} (k+1)\, u(n-k) = 2^{-n}, \quad n \in \mathbb{N}.$

Enkelsidig z-transform ger ($U = \mathcal{Z}_+ u$):

$$\left(\frac{z}{(z-1)^2} + \frac{z}{z-1} \right) U(z) = \frac{z}{z - 1/2}, \qquad \frac{z^2}{(z-1)^2} U(z) = \frac{z}{z - 1/2},$$

$$U(z) = \frac{(z-1)^2}{z(z - 1/2)} = z \frac{(z-1)^2}{z^2 (z - 1/2)} = z \left(\frac{-2}{z^2} + \frac{0}{z} + \frac{1}{z - 1/2} \right) = -\frac{2}{z} + \frac{z}{z - 1/2}, \quad |z| > \frac{1}{2}.$$

Tabell + räkneregler ger: Svar: $u(n) = -2\delta(n-1) + 2^{-n}, \quad n \in \mathbb{N}.$

3. a) $\langle e^{3t} \delta'', \varphi \rangle = \langle \delta'', e^{3t} \varphi \rangle = -\langle \delta', e^{3t} \varphi' + 3e^{3t} \varphi \rangle =$

$= \langle \delta, e^{3t} \varphi'' + 6e^{3t} \varphi' + 9e^{3t} \varphi \rangle = \varphi''(0) + 6\varphi'(0) + 9\varphi(0) =$

$= \langle \delta'' - 6\delta' + 9\delta, \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$

Svar: $\delta'' - 6\delta' + 9\delta.$

b) $(t+1)u = \delta_3 + t^2 + 1, \quad u \in \mathcal{D}'(\mathbb{R}).$

Eftersom $(t+1)\delta_3 = 4\delta_3$, och $t^2 + 1 = (t+1)(t-1) + 2$ har vi:

$(t+1)u = (t+1)\left(\frac{1}{4}\delta_3 + t - 1 + 2\,\underline{(t+1)^{-1}} \right)$, vilket ger:

Svar: $u = \frac{1}{4}\delta_3 + t - 1 + 2\,\underline{(t+1)^{-1}} + C\delta_{-1}, \quad C \in \mathbb{C}.$

4. $y''(t) - y(t) = 6\delta'(t) - 4e^{it} - 3e^{-2|t|}, \quad y \in S'.$

Fouriertransform ger: $\quad (i\omega)^2\hat{y}(\omega) - \hat{y}(\omega) = 6i\omega - 8\pi\delta(\omega-1) - \frac{12}{4+\omega^2},$

$$\hat{y}(\omega) = -\frac{6i\omega}{1+\omega^2} + \frac{8\pi}{1+\omega^2}\delta(\omega-1) + \frac{12}{(1+\omega^2)(4+\omega^2)} =$$

$$= 3\cdot\frac{-2i\omega}{1+\omega^2} + 4\pi\delta(\omega-1) + \frac{4}{1+\omega^2} + \frac{-4}{4+\omega^2}.$$

Inverstransform med tabell ger:

<u>Svar:</u> $\quad y(t) = 3e^{-|t|}\operatorname{sgn} t + 2e^{it} + 2e^{-|t|} - e^{-2|t|}.$

5. $u(t) = e^t, \; 0 \le t < 2\pi, \quad T = 2\pi$ så $\Omega = 1.$

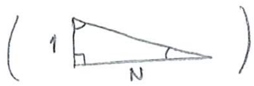
$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^t e^{-int}\,dt = \frac{1}{2\pi}\left[\frac{e^{(1-in)t}}{1-in}\right]_0^{2\pi} = \frac{e^{2\pi}-1}{2\pi(1-in)}, \quad n \in \mathbb{Z}.$$

För $N \geq 1$: $\quad \frac{1}{2\pi}\int_0^{2\pi} |u(t) - (S_N u)(t)|^2\,dt = $ / Parsevals formel /

$$= \sum_{n=-\infty}^{\infty} |(u - S_N u)^\wedge(n)|^2 = \sum_{|n|>N} |\hat{u}(n)|^2 = \sum_{|n|>N} \overbrace{\frac{(e^{2\pi}-1)^2}{4\pi^2}}^{=C}\frac{1}{(1+n^2)} =$$

$$= C\cdot 2\sum_{n=N+1}^{\infty} \frac{1}{1+n^2} \le 2C\int_N^{\infty} \frac{dx}{1+x^2} = 2C\left[\operatorname{at} x\right]_N^{\infty} = 2C\left(\frac{\pi}{2} - \operatorname{at} N\right) =$$

$$= 2C \operatorname{at}\frac{1}{N} = \underline{\underline{\frac{(e^{2\pi}-1)^2}{2\pi^2}\operatorname{at}\frac{1}{N}}}.$$



6. $u(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}, \quad t > 0.$ $\quad$ Antag först att $s > 0$.

$$(\mathcal{L}_+ u)(s) = \int_0^{\infty} \frac{\cos\sqrt{t}}{\sqrt{t}} e^{-st}\,dt = /t = r^2/ = \int_0^{\infty} \frac{\cos r}{\cancel{r}} e^{-sr^2}\cdot 2\cancel{r}\,dr =$$

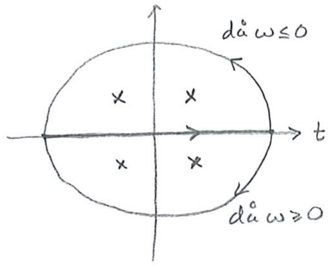
$$= 2\int_0^{\infty} e^{-sr^2}\cos r\,dr = /\text{jämn integrand}/ = \int_{-\infty}^{\infty} e^{-sr^2}\cos r\,dr = /\sin r \text{ udda}/ =$$

$$= \int_{-\infty}^{\infty} e^{-sr^2} e^{ir}\,dr = /\mathcal{F}\text{-tabell}/ = \sqrt{\frac{\pi}{s}}\, e^{-(-1)^2/4s} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}.$$

$\mathcal{L}_+ u$ är analytisk för $\operatorname{Re} s > 0$, så om vi låter $\sqrt{z}$ beteckna principalgrenen av $z^{1/2}$ får vi:

<u>Svar:</u> $\quad \sqrt{\frac{\pi}{s}}\, e^{-1/4s}, \quad \operatorname{Re} s > 0.$

7. $u(t) = \arctan t^2, \quad t \in \mathbb{R}.$



$$u'(t) = \frac{2t}{1+t^4}, \quad \text{så} \quad i\omega\hat{u}(\omega) = \int_{-\infty}^{\infty} \frac{2t\, e^{-i\omega t}}{1+t^4}\, dt.$$

Residysatsen och uppskattningar ger att

$$\int_{-\infty}^{\infty} \frac{2te^{-i\omega t}}{1+t^4}\, dt = -2\pi i \left(\operatorname*{Res}_{z=\frac{1-i}{\sqrt{2}}} + \operatorname*{Res}_{z=\frac{-1-i}{\sqrt{2}}} \right) \frac{2ze^{-i\omega z}}{1+z^4} = /\text{enkelpoler}/ \ldots$$

$$= -2\pi i\, e^{-\omega/\sqrt{2}} \sin\frac{\omega}{\sqrt{2}}, \quad \omega \ge 0.$$

u' är udda, så $i\omega\hat{u}(\omega) = -2\pi i\, e^{-|\omega|/\sqrt{2}} \sin\frac{\omega}{\sqrt{2}}$.

Detta ger att $\hat{u}(\omega) = -2\pi e^{-|\omega|/\sqrt{2}} \frac{\sin(\omega/\sqrt{2})}{\omega} + C\delta(\omega)$ för något $C \in \mathbb{C}$.

Eftersom $\arctan t^2 = \frac{\pi}{2} - \underbrace{\arctan\frac{1}{t^2}}_{\in L^1(\mathbb{R})}, \quad t \neq 0,$ fås att

$$\hat{u}(\omega) = \frac{\pi}{2}\cdot 2\pi\delta(\omega) - \underbrace{\left(\arctan\frac{1}{t^2}\right)^{\wedge}(\omega)}_{\text{kont. fkn}}, \quad \text{vilket ger att } C = \pi^2.$$

Svar: $\quad -2\pi e^{-|\omega|/\sqrt{2}} \frac{\sin(\omega/\sqrt{2})}{\omega} + \pi^2\delta(\omega).$