Lösningar, TATA77, 2025-01-08

1. $y(n+2) - y(n+1) - 2y(n) = 6$, $n \in \mathbb{N}$, $y(0) = 1$, $y(1) = 2$.

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 1z^2 - 2z - (zY(z) - 1z) - 2Y(z) = \frac{6z}{z-1},$$

$$(z^2 - z - 2)Y(z) = \frac{6z}{z-1} + z^2 + z = \frac{z^3 + 5z}{z-1}, \quad Y(z) = z\,\frac{z^2+5}{(z-2)(z+1)(z-1)} =$$

$$= z\left(\frac{3}{z-2} + \frac{1}{z+1} + \frac{-3}{z-1}\right) = \frac{3z}{z-2} + \frac{z}{z+1} - \frac{3z}{z-1}, \quad |z| > 2.$$

Tabell ger: $y(n) = 3\cdot 2^n \chi(n) + (-1)^n \chi(n) - 3\chi(n)$, $n \in \mathbb{N}$, så:

Svar: $y(n) = 3\cdot 2^n + (-1)^n - 3$, $n \in \mathbb{N}$.

2. $u(t) + \int_0^t e^r u(t-r)\,dr = 2e^{2t}\chi(t)$, $t \geq 0$.

Enkelsidig laplacetransform ger ($U = \mathcal{L}_+ u$):

$$U(s) + \frac{1}{s-1}U(s) = \frac{2}{s-2}, \quad \frac{s}{s-1}U(s) = \frac{2}{s-2}, \quad U(s) = \frac{2(s-1)}{(s-2)s} =$$

$$= \frac{1}{s-2} + \frac{1}{s}, \quad \operatorname{Re} s > 2.$$ Tabell ger $u(t) = e^{2t}\chi(t) + \chi(t)$, $t \geq 0$,

så: Svar: $u(t) = e^{2t} + 1$, $t \geq 0$.

3. a) $u(t) = \begin{cases} e^t, & t \geq 0, \\ t, & t < 0. \end{cases}$ $\quad u'(t) = \begin{Bmatrix} e^t, & t > 0 \\ 1, & t < 0 \end{Bmatrix} + (1-0)\delta(t)$,

$$u''(t) = \begin{Bmatrix} e^t, & t > 0 \\ 0, & t < 0 \end{Bmatrix} + (1-1)\delta(t) + \delta'(t).$$ Svar: $u'' = e^t \chi + \delta'$.

b) $tu = 4\delta' + 3 = t(-2\delta'' + 3\underline{t}^{-1})$ ty $t\delta'' = -2\delta'$, så:

Svar: $u = -2\delta'' + 3\underline{t}^{-1} + C\delta$, $C \in \mathbb{C}$.

c) Sätt $u = (\arctan t)\delta''(t)$. Låt $\varphi \in \mathcal{D}(\mathbb{R})$.

$$\langle u, \varphi \rangle = \langle \delta'', (\operatorname{at} t)\varphi \rangle = (-1)^2 \langle \delta, ((\operatorname{at} t)\varphi)'' \rangle =$$

$$= \langle \delta, ((\operatorname{at} t)\varphi' + \tfrac{1}{1+t^2}\varphi)' \rangle = \langle \delta, (\operatorname{at} t)\varphi'' + \tfrac{2}{1+t^2}\varphi' - \tfrac{2t}{(1+t^2)^2}\varphi \rangle =$$

$$= 0 + 2\varphi'(0) - 0 = \langle -2\delta', \varphi \rangle,$$ dvs $u = -2\delta'$.

Detta ger $\hat{u} = -2\widehat{\delta'} = -2i\omega \cdot 1$.

Svar: $-2i\omega$.

4. $u(t) = \cos t, \quad 0 \le t < \pi, \quad u(t) = 0, \quad \pi \le t < 2\pi, \quad T = 2\pi \Rightarrow \Omega = 1.$

$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{\pi} \cos t \cdot e^{-int}\,dt = \frac{1}{2\pi}\int_0^{\pi} \frac{1}{2}\left(e^{i(1-n)t} + e^{-i(1+n)t}\right)dt \overset{n \neq \pm 1}{=}$$

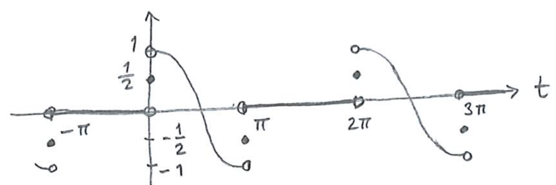
$$= \frac{1}{4\pi}\left[\frac{e^{i(1-n)t}}{i(1-n)} + \frac{e^{-i(1+n)t}}{-i(1+n)}\right]_0^{\pi} = \frac{1}{4\pi i}\left(\frac{-(-1)^n - 1}{1-n} - \frac{-(-1)^n - 1}{1+n}\right) =$$

$$= \frac{1}{4\pi i}\,\frac{2n(-(-1)^n - 1)}{1-n^2} = \frac{(1+(-1)^n)n}{2\pi i(n^2-1)}, \quad n \neq \pm 1.$$

$$\hat{u}(1) = \frac{1}{4\pi}\int_0^{\pi}(1 + e^{-i2t})\,dt = \frac{1}{4}, \quad \hat{u}(-1) = \frac{1}{4\pi}\int_0^{\pi}(e^{i2t} + 1)\,dt = \frac{1}{4}.$$

Delsvar: $\displaystyle \frac{1}{4}e^{it} + \frac{1}{4}e^{-it} + \sum_{n \neq \pm 1} \frac{(1+(-1)^n)n}{2\pi i(n^2-1)} e^{int}.$

Fourierseriens summa, enligt satsen om punktvis konvergens:



5. $y''(t) - 3y'(t) + 2y(t) = a\delta(t-b), \quad t \ge 0, \quad y(0) = 1, \quad y'(0) = 0.$

$\mathcal{L}_+$ ger $(Y = \mathcal{L}_+ y)$: $s^2 Y(s) - 1s - 0 - 3(sY(s) - 1) + 2Y(s) = ae^{-bs}$

(då $b \ge 0$, men om $b < 0$ blir $\mathcal{L}_+(HL) = 0$ och $y(t)$ blir inte konstant),

$$(s^2 - 3s + 2)Y(s) = s - 3 + ae^{-bs}, \quad Y(s) = \frac{s-3}{(s-2)(s-1)} + ae^{-bs}\frac{1}{(s-2)(s-1)} =$$

$$= \frac{-1}{s-2} + \frac{2}{s-1} + ae^{-bs}\left(\frac{1}{s-2} - \frac{1}{s-1}\right), \quad \operatorname{Re} s > 2.$$

Så $y(t) = -e^{2t} + 2e^{t} + a(e^{2(t-b)} - e^{t-b})\chi(t-b), \quad t \ge 0.$

Om $t > b$ fås $y(t) = (-1 + ae^{-2b})e^{2t} + (2 - ae^{-b})e^{t}$ och detta är

konstant omm $-1 + ae^{-2b} = 0$ och $2 - ae^{-b} = 0$, dvs

$-1 + 2e^{-b} = 0$ och $ae^{-b} = 2$, dvs $b = \ln 2$ och $a = 4$.

Svar: $a = 4, \ b = \ln 2.$

6. $$n u(n) + \sum_{k=0}^{n} (2^{n-k}+1)\, u(k) = 2\delta(n) + 2^n, \quad n \in \mathbb{N}.$$

$\mathcal{Z}_+$ ger ($U = \mathcal{Z}_+ u$): $$-zU'(z) + \left(\frac{z}{z-2} + \frac{z}{z-1}\right)U(z) = 2 + \frac{z}{z-2},$$

$$U'(z) - \left(\frac{1}{z-2} + \frac{1}{z-1}\right)U(z) = -\frac{3z-4}{(z-2)z}.$$

Int. fakt. $\dfrac{1}{(z-2)(z-1)}$ (från $e^{-\log(z-2)-\log(z-1)}$) ger:

$$\left(\frac{1}{(z-2)(z-1)}U(z)\right)' = -\frac{3z-4}{(z-2)^2(z-1)z} = \frac{-1}{(z-2)^2} + \frac{0}{z-2} + \frac{1}{z-1} + \frac{-1}{z} =$$

$$= \left(\frac{1}{z-2} - \mathrm{Log}\frac{z}{z-1}\right)',$$

$$\frac{1}{(z-2)(z-1)}U(z) = \frac{1}{z-2} - \mathrm{Log}\frac{z}{z-1} + C, \quad \text{ngt } C \in \mathbb{C}.$$

$|z| \to \infty$ ger $0 \cdot u(0) = 0 - 0 + C$, dvs $C = 0$

(ty $U(z) = \sum_{n=0}^{\infty} u(n) z^{-n}$, så $U(z) \to u(0)$ då $|z| \to \infty$).

Alltså fås $$U(z) = z - 1 - (z-2)(z-1)\,\mathrm{Log}\frac{z}{z-1} =$$

$$= z - 1 - (z^2 - 3z + 2)\left(\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots\right), \quad |z| > 1.$$

Avläsning av koefficienter ger:

z^1: 0 (följer också av att $U = \mathcal{Z}_+ u$),

z^0: $-1 - \frac{1}{2} + 3 = \frac{3}{2}$,

z^{-n}, $n > 0$: $$-\frac{1}{n+2} + 3\frac{1}{n+1} - 2\frac{1}{n} = -\frac{n+4}{(n+2)(n+1)n}.$$

<u>Svar:</u> $$u(n) = \begin{cases} 3/2, & n = 0, \\ -\dfrac{n+4}{(n+2)(n+1)n}, & n > 0. \end{cases}$$

7. $u(t) = \ln|t|$, $t \neq 0$.

Sätt $v(t) = (\ln t)\chi(t)$. Då är $u(t) = v(t) + v(-t)$.

Tabell: $(\mathcal{L}v)(s) = -\frac{\gamma + \operatorname{Log} s}{s}$, Re $s > 0$, så för $\varphi \in \mathcal{H}$:

$$\langle (\mathcal{L}v)(s), \varphi(s) \rangle = \int_L -\frac{\gamma + \operatorname{Log} s}{s} \varphi(s) \frac{ds}{i} \qquad (L:\ s = 1 + i\omega,\ \omega: -\infty \to \infty)$$

$$= \int_L \left(\gamma \operatorname{Log} s + \frac{1}{2} \operatorname{Log}^2 s\right) \varphi'(s) \frac{ds}{i} \qquad (\text{Genom part. int.})$$

/ Enbart logaritmisk singularitet, så L kan ersättas med $s = i\omega$, $\omega: -\infty \to \infty$ (Cauchys integralsats). /

$$= \int_{-\infty}^{\infty} \left(\gamma\left(\ln|\omega| + \frac{i\pi}{2} \operatorname{sgn} \omega\right) + \frac{1}{2}\left(\ln|\omega| + \frac{i\pi}{2} \operatorname{sgn} \omega\right)^2\right) \varphi'(i\omega)\, d\omega$$

/ $\varphi_0(\omega) = \varphi(i\omega)$ ger $\varphi_0'(\omega) = i\varphi'(i\omega)$ /

$$= \int_{-\infty}^{\infty} \left(\gamma \ln|\omega| + \frac{i\pi\gamma}{2} \operatorname{sgn} \omega + \frac{1}{2} \ln^2|\omega| + \frac{i\pi}{2} (\operatorname{sgn} \omega) \ln|\omega| - \frac{\pi^2}{8}\right)(-i)\varphi_0'(\omega)\, d\omega.$$

Derivering ger nu att

$$(\mathcal{F}v)(\omega) = i\gamma\, \underline{\omega^{-1}} - \pi\gamma\,\delta(\omega) + \frac{i}{2}\left(\ln^2|\omega|\right)' - \frac{\pi}{2}\left((\operatorname{sgn} \omega) \ln|\omega|\right)'.$$

Detta ger (udda, jämna):

<u>Svar:</u> $\hat{u}(\omega) = -2\pi\gamma\,\delta(\omega) - \pi\left((\operatorname{sgn} \omega) \ln|\omega|\right)'.$