Written Examination in Discrete Mathematics TATA82, TEN1, 2023–06–03, kl 14–19.

No calculator. Complete motivations required.

For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

1. Show using mathematical induction that $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$, for all $n \geq 1$.

2. A simple planar, connected graph G has nodes of degrees 3 and 6 (no other degree) and 10, 11 or 12 faces. It contains at least 3 nodes of each degree. Determine the number of nodes and edges in all possible graphs satisfying the conditions. (3p)

3. (a) Provide a cheapest Prefix Code to encode the message DIGITAL TEST. Encode the message with the Prefix code you have obtained. (1p)

   (b) Solve the system of linear modular equations

$$\begin{cases} 5x & +3y & -2z & \equiv & 12 \bmod 23 \\ 3x & -5y & +z & \equiv & 21 \bmod 23 \\ 6x & +4y & +3z & \equiv & 3 \bmod 23 \end{cases} \qquad (2p)$$

4. (a) Consider the set of matrices $\mathcal{M} = \{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};$ with $a_{ij} = 0$ or $1\}$. How many funktions $f : \mathcal{M} \to \{0, 1\}$ are there? (1p)

   (b) Consider the alphabet with the symbols $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, X, a, b, c, d, e, f\}$. How many words of length 10 are there with the conditions that the words cannot begin with a digit and not finish with a letter? (2p)

5. Solve the recurrence equation $a_{n+1} - a_n = n^3 + 6n^2 + 11n + 6,\ n \geq 0,\ a_0 = 0$ (3p)

6. An exam is constructed so that a student should match 8 questions with 8 given answers. In how many ways can the student answer so that every question gets the wrong answer? Answer with an integer. (3p)

7. Consider the set of matrices $\mathcal{M} = \{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};$ with $a_{ij} = 0$ or $1\}$. We define a relation $\preceq$ on $\mathcal{M}$ by: $A = (a_{ij}) \preceq B = (b_{ij})$ if for every entry $a_{ij} \leq b_{ij}$.

   (a) Show that $\preceq$ is a partial ordering. (1p)

   (b) Determine lub$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ and glb$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$. (1p)

   (c) Determine, if they exist, the greatest and least elements in $(\mathcal{M}, \preceq)$ (1p)

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Tentamen i Diskret Matematik, TATA82, TEN1, 2023–06–03, kl 14–19.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

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1. Visa med induktionsprincipen att $\displaystyle\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$, för alla $n \geq 1$.

2. En enkel, planär, sammanhängande graf G har noder of gradtal 3 och 6 (inget annat gradtal) och 10, 11 eller 12 regioner. Grafen innehåller minst 3 noder av varje gradtal. Bestäm antalen noder och kanter i alla möjliga grafer som uppfyller villkoren. (3p)

3. (a) Ange en billigaste Prefixkod för att koda meddelandet DIGITAL TEST. Koda meddelandet med prefixkoden du har fått. (1p)

   (b) Lös systemet av modulära ekvationer

$$\begin{cases} 5x & +3y & -2z & \equiv & 12 \bmod 23 \\ 3x & -5y & +z & \equiv & 21 \bmod 23 \\ 6x & +4y & +3z & \equiv & 3 \bmod 23 \end{cases} \qquad (2p)$$

4. (a) Betrakta mängden av matriser $\mathcal{M} = \{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$; med $a_{ij} = 0$ eller $1\}$. Hur många funktioner $f : \mathcal{M} \to \{0,1\}$ finns det ? (1p)

   (b) Betrakta alfabetet med symbolerna $\mathcal{A} = \{1,2,3,4,5,6,7,8,9,X,a,b,c,d,e,f\}$. Hur många ord av längd 10 finns det med villkoren att ordet inte kan börja med en siffra och inte sluta med en bokstav? (2p)

5. Lös den rekursiva ekvationen $a_{n+1} - a_n = n^3 + 6n^2 + 11n + 6$, $n \geq 0$, $a_0 = 0$ (3p)

6. En tentamen är konstruerad så att 8 frågor ska paras ihop med 8 givna svar. På hur många sätt kan det göras så att man får fel svar på varje fråga? Svara med heltal. (3p)

7. Betrakta mängden av matriser $\mathcal{M} = \{A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$; med $a_{ij} = 0$ eller $1\}$. Vi definierar en relation $\preceq$ på $\mathcal{M}$ genom: $A = (a_{ij}) \preceq B = (b_{ij})$ om för varje element $a_{ij} \leq b_{ij}$.

   (a) Visa att $(\mathcal{M}, \preceq)$ är en pomängd. (1p)

   (b) Bestäm $\text{lub}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ och $\text{glb}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$. (1p)

   (c) Bestäm, om de existerar, största och minsta element i $(\mathcal{M}, \preceq)$ (1p)

Answers TATA82 Discrete Mathematics 3/6 2023

1) Show with IP that $\sum_{1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ $\forall n \geq 1$

i) First we controll the formula for $n=1$

$$(1)(2)(3) = \frac{(1)(2)(3)(4)}{4} \quad \text{. true}$$

ii) Secondly, assume that $\sum_{1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

for some $n \geq 1$. We see that

$$\sum_{1}^{n+1} k(k+1)(k+2) = \sum_{1}^{n} k(k+1)(k+2) + (n+1)(n+2)(n+3) =$$

$$\overset{Ant}{=} \frac{n(n+1)(n+2)(n+3)}{4} + (n+1)(n+2)(n+3) =$$

$$= (n+1)(n+2)(n+3)\left(\frac{n}{4}+1\right) = \frac{(n+1)(n+2)(n+3)(n+4)}{4}$$

as desired

2) A simple, planar, connected graph has nodes of degrees 3 and 6 and 10, 11 or 12 faces. Determine the number of nodes and edges of such a graph if it contains at least 3 nodes of every degree (3 or 6)

We have using Hand-Shaking Lemma and Euler's formula: x and y are the number of nodes of deg 3 and 6

$$(1)\begin{cases} 3x+6y = 2e \\ x+y-e+10 = 2 \end{cases} \quad (2)\begin{cases} 3x+6y = 2e \\ x+y+11-e = 2 \end{cases} \quad \begin{cases} 3x+6y = 2e \\ x+y+12-e = 2 \end{cases}$$

$$(1)\begin{cases} 3x+6y = 2e \\ x+4y = 16 \end{cases} \quad (2)\begin{cases} 3x+6y = 2e \\ x+4y = 18 \end{cases} \quad (3)\begin{cases} 3x+6y = 2e \\ x+4y = 20 \end{cases}$$

Solving the Diophantine equations $\begin{cases} x+4y=16 \\ x+4y=18 \\ x+4y=20 \end{cases}, x,y \geq 3$

and as the number of nodes of degree 3 is EVEN we get (1) $f = 10$, $n = x+y = 4+3$, $e = 15$

(2) $f = 11$ $\quad n = x+y = 6+3 = 9$, $e = 18$

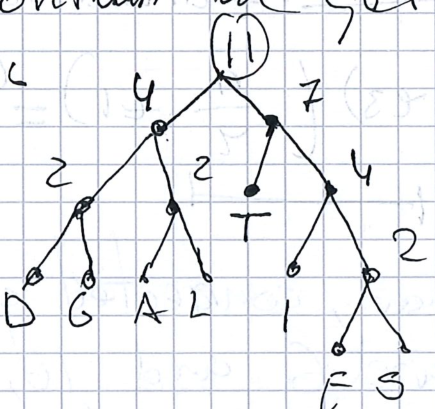
(3a) $f = 12$ $\quad n = x_{(3)} + y_{(3)} = 8+3 = 11$, $e = 21$

(3b) $f = 12$ $\quad n = x_{(4)} + y_{(4)} = 4+4 = 8$, $e = 18$

3a) DIGITAL TEST has as forest

1 D, 2 I, 1 G, 3 T, 1 A, 2 L, 1 E, 1 S. with

Hoffman's algorithm we get a complete binary tree:



Code: $D \equiv 000$, $I \equiv 110$, $G \equiv 001$, $T \equiv 10$, $A \equiv 010$, $L \equiv 011$, $E \equiv 1110$, $S \equiv 1111$

Message 000110 001110 10010 01110 1110 1111 10

3b)
$$\begin{cases} 5x+3y-2z \equiv 12 \bmod 23 \\ 3x-5y+z \equiv 21 \bmod 23 \\ 6x+4y+3z \equiv 3 \bmod 23 \end{cases} \Leftrightarrow \begin{cases} 3x-5y+z \equiv 21 \\ 14y+z \equiv 7 \\ 12y+11z \equiv 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3x-5y+z \equiv 21 \\ 14y+z \equiv 7 \\ 2z \equiv 4 \end{cases} . \text{ So } \begin{cases} y \equiv (5)(5) \equiv 2 \\ z \equiv 2 \\ x \equiv (8)(6) \equiv 2 \end{cases}$$

4a) $M = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; a_{ij} = 0 \text{ or } 1 \right\}$ $\quad f : M \to \{0,1\}$

$|M| = 2^4 = 16$. No of functions is $2^{16} = 65536$

4B) Again multiplication principle

Alt1: X a letter $(7)(16)^8(9)$

Alt2: X a digit $(6)(16)^8(10)$

5) Solve the recurrence eq. $a_{n+1} - a_n = n^3 + 6n^2 + 11n + 6$

$a_0 = 0$. Ch. Eq. $r - 1 = 0$, $r_1 = 1$, $m_1 = 1$

$a_n^{(h)} = A$

For the $a_n^{(p)} = (b_0 + b_1 n + b_2 n^2 + b_3 n^3) n$. Setting it in the eq

$$b_0 (n+1) + b_1 (n+1)^2 + b_2 (n+1)^3 + b_3 (n+1)^4 - b_0 n - b_1 n^2 - b_2 n^3 - b_3 n^4 = n^3 + 6n^2 + 11n + 6$$

We get
$$\begin{cases} b_3 (1-1) = 0 \\ 4b_3 + b_2 - b_2 = 1 \\ 6b_3 + 3b_2 = 6 \\ 4b_3 + 3b_2 + 2b_1 + b_0 - b_0 = 11 \\ b_3 + b_2 + b_1 + b_0 = 6 \end{cases} \Rightarrow \begin{cases} b_3 = \frac{1}{4} \\ b_2 = \frac{6}{4} \\ b_1 = \frac{11}{4} \\ b_0 = \frac{6}{4} \end{cases}$$

$$a_n^{(p)} = \frac{n^4 + 6n^3 + 11n^2 + 6n}{4}$$

As $a_0 = 0$, $A = 0$ $\quad a_n = \frac{n^4 + 6n^3 + 11n^2 + 6n}{4} = \frac{n(n+1)(n+2)(n+3)}{4}$

Question 5) and 1) look for the same !!

6) Question 6) wants to set 8 answers on 6 boxes such that not answer get the box with the same number, i.e we look for the number of derangements on 8 symbols:

$$\boxed{d_8 = \sum_{i=2}^{8} \frac{(-1)^i\, 8!}{i!}} = 20160 - 6720 + 1680 - 336 + 56 - 8 + 1 = \underline{14833}$$

7) Consider $M = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; a_{ij} = 0 \text{ or } 1 \right\}$

We define $\preceq$ by : $A = (a_{ij}) \preceq B = (b_{ij})$ if $\forall i,j$ $a_{ij} \le b_{ij}$

a) $(M, \preceq)$ a poset since

i) $\forall A \in M$ $\quad a_{ij} \le a_{ij}$ and $A \preceq A$

ii) If $A \preceq B$ and $B \preceq A$, then $\forall i,j$
$a_{ij} \le b_{ij}$ and $b_{ij} \le a_{ij}$ so $a_{ij} = b_{ij}$
and $A = B$

iii) If $A \preceq B$ an $B \preceq C = (c_{ij})$ we have
$(\forall (i,j))$ $a_{ij} \le b_{ij}$, $b_{ij} \le c_{ij}$ and $a_{ij} \le c_{ij}$
$A \preceq C$

b) $\text{lub}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\} = \begin{pmatrix}1&1\\1&1\end{pmatrix}$

$\text{glb}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

c) the greatest element: $\begin{pmatrix}1&1\\1&1\end{pmatrix}$

the least element: $\begin{pmatrix}0&0\\0&0\end{pmatrix}$