LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA82, TEN1, 2023–08–17, kl 08–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

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1. Betrakta Fibonaccital som definieras av $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}, n \geq 3$, vi kan anta att $f_0 = 0$. Visa med induktionsprincipen att $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$, för alla $n \geq 3$ (Observera att räknare n har två olika meningar).

2. Betrakta de fullständiga binära träd F_n som definieras som följer: F_1 och F_2 är de isolerade noderna, och för $n \geq 3$ fås F_n genom att sammanfoga F_{n-1} och F_{n-2} till en ny rot.

   (a) Visa att antal löv i F_n är n:te Fibonaccitalet. (2p)

   (b) Vad är höjden av F_n? (1p)

3. (a) Betrakta den fullständiga bipartita grafen $K_{n,2}$. Hur många uppspännande träd har $K_{n,2}$? (1p)

   (b) På hur många sätt kan man dela 24 personer i 6 grupper, med 4 i varje, om ordningen i vilken grupperna ligger är irrelevant? (1p)

   (c) Hur många ekvivalensrelationer kan man definiera på en mängd med 24 element om varje av de 6 ekvivalensklasserna innehåller exakt 4 element? (1p)

4. (a) Hur många heltal $n \geq 2$ uppfyller att $200 \equiv 20 \bmod n$? (1p)

   (b) Bestäm $99^{841} + 100^{841} + 799^{841} + 800^{841} \bmod 899$ (2p)

5. Visa att funktionen $F(n) = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \ldots$, $n = 1, 2, \ldots$, där termer där nämnaren blir större än täljaren inte räknas, uppfyller den rekursiva ekvationen för Fibonaccital och således är $F(n)$ n:te Fibonaccitalet.

6. En affär har (i affären) dockor i sex modeller: 8 av modell A, 8 av modell B, 8 av modell C, 7 av modell D, 7 av modell E och 7 av modell F. Hur många olika presenter som innehåller 40 dockor kan man göra till en förskola?

7. Betrakta ett givet **fullständigt** rotat binärt träd $\mathcal{T}$ med rot r och höjden 2^{64}. Vi definierar en relation $\mathcal{R}$ på $(\mathcal{T}, r)$ genom: $v_1 \mathcal{R} v_2$ om v_1 ligger i vägen från r till v_2.

   (a) Visa att $\mathcal{R}$ är en partialordning. (1p)

   (b) Bestäm minsta och största element. (1p)

   (c) Är $(\mathcal{T}, \mathcal{R})$ en lattice? (1p)

Written Examination in Discrete Mathematics TATA82, TEN1, 2023–08–17, kl 08–13.
No calculator. Complete motivations required.
For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

1. Consider Fibonacci numbers, defined by $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}, n \geq 3$, we can assume that $f_0 = 0$. Show using mathematical induction that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$, for all $n \geq 1$ (Observe that the counter n is used in two different ways).

2. Consider the complete binary trees F_n defined as follows: F_1 and F_2 are the isolated nodes, and for $n \geq 3$ one gets F_n by merging F_{n-1} and F_{n-2} into a new root.
   - (a) Show that the number of leaves of F_n is the n:th Fibonacci number. (2p)
   - (b) What is the height of F_n? (1p)

3. (a) Consider the complete bipartite graph $K_{n,2}$. How many spanning trees does $K_{n,2}$ have? (1p)
   - (b) In how many ways can one divide 24 people in 6 groups, with 4 in each group, where the order of the groups is irrelevant? (1p)
   - (c) How many equivalence relations can one define on a set with 24 elements if each of the 6 equivalence classes contains exactly 4 elements? (1p)

4. (a) How many integers $n \geq 2$ do satisfy that $200 \equiv 20 \bmod n$? (1p)
   - (b) Determine $99^{841} + 100^{841} + 799^{841} + 800^{841}$ mod 899 (2p)

5. Show that the function $F(n) = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \ldots$, $n = 1, 2, \ldots$, where terms with denominator larger than numerator are **not** counted, satisfies the recursive equation for Fibonacci numbers and so $F(n)$ is the n:th Fibonacci number.

6. A shop has on location dolls in six models: 8 in model A, 8 in B, 8 in C, 7 in model D, 7 in E and 7 in F. How many different gifts containing 40 dolls can one give to a play school?

7. Consider a **complete** binary tree $\mathcal{T}$ with root r and height 2^{64}. We define a relation $\mathcal{R}$ on $(\mathcal{T}, r)$ as follows: $v_1 \mathcal{R} v_2$ if v_1 is an ancestor to v_2, or v_2 itself.
   - (a) Show that $\mathcal{R}$ is a partial ordering. (1p)
   - (b) Determine the least and greatest elements. (1p)
   - (c) Is $(\mathcal{T}, \mathcal{R})$ a lattice? (1p)

Discrete Mathematics TATA 82 17/8 2023

Answers:

1) Show using IP that $\sum_{k=1}^{2n} f_{k-1} f_k = f_{2n}^2 \quad \forall n \geq 1$

where f_i is the i:th Fibonacci number ($f_0 = 0$)

1) We show that it is true for $n = 1$

$f_0 f_1 + f_1 f_2 = (0)(1) + (1)(1) = (1)^2 = f_2^2$. True

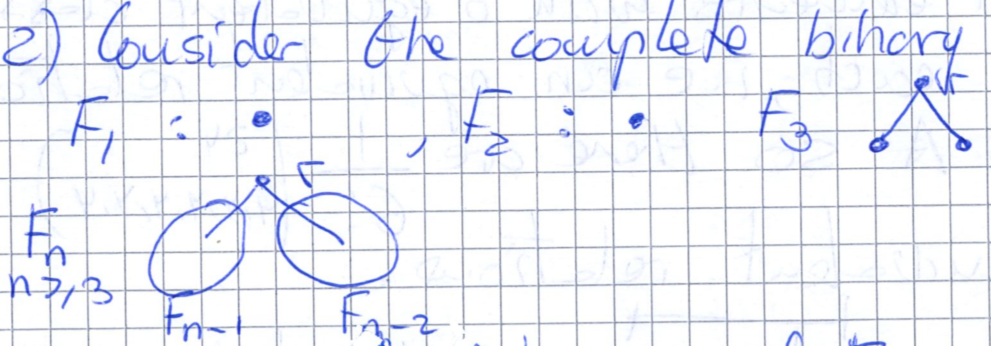
2) Assuming that the formula is true for any p between 1 and $n \geq 1$ we check for $n+1$: $f_0 f_1 + \dots + f_{2n-1} f_{2n} + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2}$

$\overset{\text{IP (assump)}}{=} f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} =$

$= f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2n+2} =$ (Fibonacci nos)

$= (f_{2n} + f_{2n+1}) f_{2n+2} \overset{\text{Fibonacci nos}}{=} \underline{f_{2n+2}^2}$ as required

2) Consider the complete binary tree



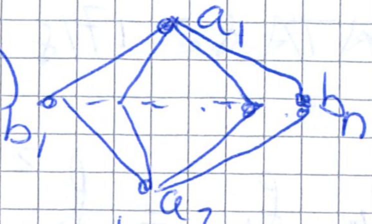
the number of leaves of F_n is $\begin{cases} n=1 & l_1 = 1 \text{ (isolated)} \\ n=2 & l_2 = 1 \text{ (isolated)} \\ n \geq 3 & l_n = l_{n-1} + l_{n-2} \end{cases}$

since the leaves of F_n are the leaves of F_{n-1} plus the ones in F_{n-2}. As l_n follows the equation for Fibonacci number $\underline{l_n = f_n}$ as in 1.

3) As we get $F_{n \geq 3}$ by merging F_{n-1} and F_{n-2} to a root the height of F_n increases by 1 at each step $n \geq 3$

$\underline{\text{height}(F_1) = \text{height}(F_2) = 0}$

$\underline{\text{height}(F_n) = n-2, \ n \geq 3}$

3a) 

Each spanning tree contains $n+2-1 = n+1$ edges

One vertex type b is connected to both a_1 and a_2: n choices

the other $n-1$ vertices type b are joined to a_1 or to a_2 : $(2)^{n-1}$ choices

Totally $\underline{n(2)^{n-1} \text{ spanning trees}}$

3b) First the order of the people in each group is irrelevant : $\binom{24}{4,4,4,4,4,4}$

As the order of the groups is irrelevant we get $\dfrac{1}{6!}\binom{24}{4,4,4,4,4,4}$

3c) Each division in 3b) is a partition of a set A of 24 elements with 6 equivalent classes of order 4 each, i.e an equivalent relation on the set A so there are $\dfrac{1}{6!}\binom{24}{4,4,4,4,4,4}$ such equivalent relations.

4a) How many integers $n \geq 2$ do satisfy $200 \equiv 20 \mod n$

That is $200-20 = kn$ with $k \in \mathbb{Z}^+$

$\Longleftrightarrow 180 = kn$ $k \in \mathbb{Z}^+$, n is a divisor of $180 \geq 2$

As $180 = (2)^2(3)^2(5)$, there are $\underline{(3)(3)(2)-1 = 17 \text{ such } n \geq 2}$

4b) $x \equiv 99^{841} + 100^{841} + 799^{841} + 800^{841} \mod 899$

$899 = (31)(29)$ and $\varphi(899) = 840$ so

$99^{841} + 100^{841} + 799^{841} + 800^{841} \overset{\text{Eulers th.}}{\equiv} 99 + 100 + (-100) + (-99) \equiv 0 \mod 899$

5) The recursive equation for
$$F(n) = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \dots \quad \text{where}$$
$\binom{n-i}{j} = 0$ if $j > n-i$ is:

$n=1$ $F(1) = \binom{0}{0} = 1$ | $n=2$ $F(2) = \binom{1}{0} = 1$

for $n = 3$ $F(3) = \binom{2}{0} + \binom{1}{1} = \binom{0}{0} + \binom{1}{0} = F(1)+F(2)$

In general for $n \geq 3$
$$F(n) = \begin{cases} \binom{n-1}{0} + \binom{n-2}{1} + \dots + \binom{n-i}{n-i} & n \text{ odd} \\ \binom{n-1}{0} + \binom{n-2}{1} + \dots + \binom{n-i}{n-i-1} & n \text{ even} \end{cases}$$
(By the way $i = \lfloor \frac{n}{2} \rfloor$)

Using that $\binom{n-1}{0} = \binom{n-1-1}{0} = 1$, that $\binom{n-i}{n-i} = \binom{n-1-i}{n-1-i} = 1$ and for the remainder terms $\binom{n-j}{k} = \binom{n-1-j}{k} + \binom{n-1-j}{k-1}$
$$1 \leq k \leq n-j-1$$
we get
$$F(n) = \binom{n-1}{0} + \binom{n-2}{1} + \dots + \binom{n-i}{n-i} \quad n \text{ odd}$$
$$= \binom{(n-1)-1}{0} + \binom{(n-1)-2}{1} + \dots + \binom{n-1-i}{n-2-i} + \binom{(n-2)-1}{0} + \dots \binom{(n-2)-i}{(n-2)-i}$$
$$= F(n-1) + F(n-2)$$
n even $F(n) = \binom{n-1}{0} + \dots + \binom{n-i}{n-i-1} =$
$$= \binom{n-2}{0} + \binom{(n-1)-2}{1} + \dots + \binom{n-1-i}{n-1-i} + \binom{(n-2)-1}{0} + \dots + \binom{n-2-i}{n-3-i}$$
$$= F(n-1) + F(n-2) \quad \text{. As required}$$

6) We look for the number of solutions $x_A + x_B + x_C + x_D + x_E + x_F = 40$ where

x_A, x_B, x_C integers $0 \le x_A, x_B, x_C \le 8$
and x_D, x_E, x_F integers $0 \le x_E, x_D, x_F \le 7$,
with $x_A, x_B, x_C, x_D, x_E, x_F$ are the number of dolls of respective type.

Consider $A_1 = \{ \text{solut where } x_A \ge 9 \}$
$A_2 = \{ \text{sol} ; x_B \ge 9 \}$, $A_3 = \{ \text{sol} ; x_C \ge 9 \}$
$A_4 = \{ \text{sol} ; x_D \ge 8 \}$, $A_5 = \{ \text{sol}, x_E \ge 8 \}$, $A_6 = \{ \text{sol} ; x_F \ge 8 \}$

We want $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6}| = |\mathcal{U}| - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6|$

i) $|\mathcal{U}| = \binom{40+5}{5} = \binom{45}{5}$

ii) $|A_1| = |A_2| = |A_3| = \binom{45-9}{5}$; $|A_4| = |A_5| = |A_6| = \binom{45-8}{5}$

iii) $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = \binom{45-18}{5}$; $\underset{4 \le i < j \le 6}{|A_i \cap A_j|} = \binom{45-16}{5}$

$\underset{1 \le i \le 3,\ 4 \le j \le 6}{|A_i \cap A_j|} = \binom{45-17}{5}$

iv) $|A_1 \cap A_2 \cap A_3| = \binom{45-27}{5}$; $|A_4 \cap A_5 \cap A_6| = \binom{45-24}{5}$

$\underset{1 \le i_1 < i_2 \le 3,\ 4 \le j}{|A_{i_1} \cap A_{i_2} \cap A_j|} = \binom{45-26}{5}$; $\underset{i \le 3,\ 4 \le j_1 < j_2 \le 6}{|A_i \cap A_{j_1} \cap A_{j_2}|} = \binom{45-25}{5}$

v) $\underset{4 \le j}{|A_1 \cap A_2 \cap A_3 \cap A_j|} = \binom{45-35}{5}$; $\underset{i \le 3}{|A_i \cap A_4 \cap A_5 \cap A_6|} = \binom{45-33}{5}$

$\underset{1 \le i_1 < i_2 \le 3,\ 4 \le j_1 < j_2 \le 6}{|A_{i_1} \cap A_{i_2} \cap A_{j_1} \cap A_{j_2}|} = \binom{45-34}{5}$

vi) $|A_1 \cap A_2 \cap A_3 \cap A_{j_1} \cap A_{j_2}| = 0$; $|A_{i_1} \cap A_{i_2} \cap A_4 \cap A_5 \cap A_6|$

Totally:

$\binom{45}{5} - 3\binom{36}{5} - 3\binom{37}{5} + 3\binom{27}{5} + 3\binom{29}{5} + 9\binom{28}{5} - \binom{18}{5} - \binom{21}{5} - 9\binom{19}{5} - 9\binom{20}{5} + 3\binom{10}{5} + 3\binom{12}{5} + 9\binom{11}{5}$

7) Consider a complete binary tree T with root r and height 2^{64}. We define a relation $\mathcal{R}$ on (T, r) by $v_1 \mathcal{R} v_2$ if v_1 ancestor to v_2 or $v_1 = v_2$

a) $\mathcal{R}$ is a partial ordering since:

i) $\mathcal{R}$ is reflexive since for every vertex v
$v \mathcal{R} v$ since $v = v$

ii) $\mathcal{R}$ antisymmetric: if $v_1 \mathcal{R} v_2$ and $v_2 \mathcal{R} v_1$ then, as v_1 and v_2 cannot be ancestor to each other, $v_1 = v_2$

iii) $\mathcal{R}$ transitive: if $v_1 \mathcal{R} v_2$ and $v_2 \mathcal{R} v_3$ v_1 is on the [illegible] from r to v_2 and v_2 is on the path from r to v_3 so v_1 is on the path from r to v_3 (the paths are unique) and $v_1 \mathcal{R} v_3$

b) the least element is r (the common ancestor). There is no greatest element since we have at least 2 leaves and leaves are <u>not</u> comparable

c) It is not a lattice since if l_1 and l_2 are leaves <u>there is not lub(l_1, l_2)</u>