LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA82, TEN1, 2022–06–03, kl 14–19.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

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1. Visa med induktionsprincipen att $\sum_{k=2}^{n}(k+1)^3 = \frac{n^4+6n^3+13n^2+12n-32}{4}$, för alla $n \geq 2$.

2. Är grafen G nedan hamiltonsk? ~~planär? bipartit?~~ Bestäm dess kromatiska tal $\chi(G)$.

3. (a) Ange en billigast Prefixkod för att koda meddelandet GENERAL KAFFEMATTE. Koda meddelandet med prefixkoden du har fått. (2p)

   (b) Ange den binära representationen av heltalet 10537 (1p)

4. (a) Lös ekvationen $x^3 - x^2 + 3x \equiv 6 \text{mod} 11$. (2p)

   (b) Bestäm antalet positiva heltal mellan 1 och 5005 som är relativt prima med 5005 (1p)

5. Lös systemet av rekursiva ekvationer

$$\begin{cases} a_{n+1} = 2a_n - b_n + (n+1) \\ b_{n+1} = -2a_n + 3b_n + (n)^2 \end{cases} \quad \text{med} \quad a_0 = 2, \quad b_0 = 2.$$

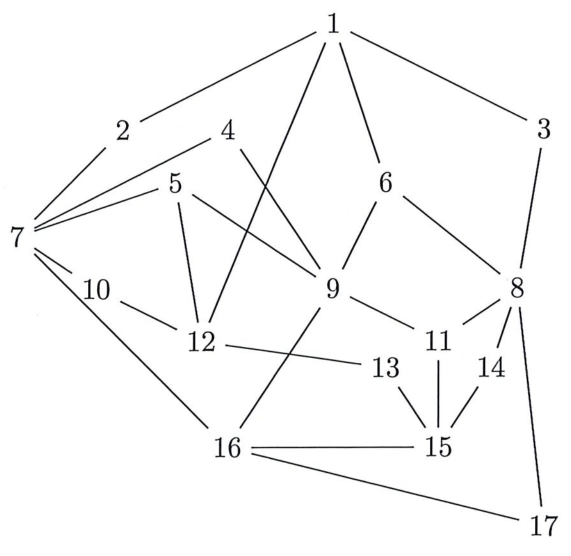
6. Sex studenter kommer till en digital tentamen med laptop, tablet och mobil. De måste lämna alla apparater i ett låst skåp. Om apparaterna lämnas ut slumpmässigt, vad är sannolikheten att ingen student får tillbaka en enda av sina apparater? Skriv svar som en produkt av rationella tal.

7. Betrakta heltalen $\mathbb{Z}$. Vi definierar en relation $\preceq$ på $\mathbb{Z}$ som: $a \preceq b$ om ett av följande villkor uppfylls: **i)** $a = b$ eller **ii)** $|a| < |b|$.

   (a) Visa att $\preceq$ är en partialordning. (1p)

   (b) Visa att $\preceq$ inte är en totalordning. (1p)

   (c) Bestäm, om de existerar, möb$\{5, -5\}$ och sub$\{5, -5\}$ (1p)



$G:$

Written Examination in Discrete Mathematics TATA82, TEN1, 2022–06–03, kl 14–19.
No calculator. Complete motivations required.
For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

1. Show using mathematical induction that $\sum_{k=2}^{n}(k+1)^3 = \dfrac{n^4+6n^3+13n^2+12n-32}{4}$, for all $n \geq 2$.

2. Consider the graph G below: Is it Hamiltonian? planar ? bipartite ? Determine its chromatic number $\chi(G)$.

3. (a) Provide a cheapest Prefix Code to encode the message GENERAL KAFFEMATTE. Encode the message with the Prefix code you have obtained. (2p)
   (b) Give the binary representation of the integer 10537 (1p)

4. (a) Solve the equation $x^3 - x^2 + 3x \equiv 6 \bmod 11$. (2p)
   (b) Determine the number of positive integers between 1 and 5005 that are relative prime with 5005 (1p)

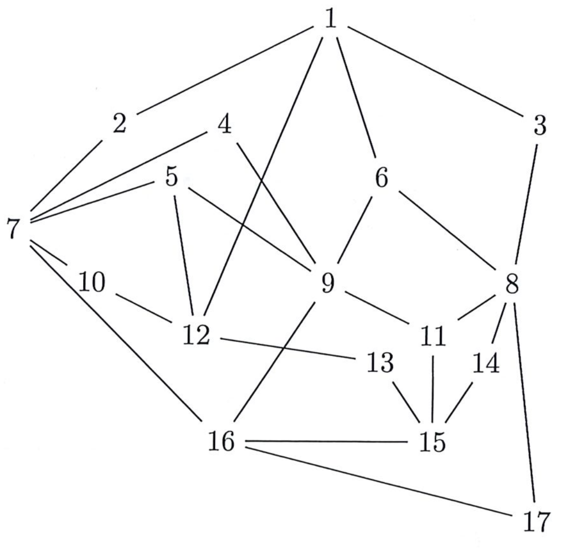
5. Solve the system of recurrence equations:

$$\begin{cases} a_{n+1} = 2a_n - b_n + (n+1) \\ b_{n+1} = -2a_n + 3b_n + (n)^2 \end{cases} \quad \text{with} \quad a_0 = 2, \quad b_0 = 2.$$

6. Six students come to a digital examination with laptop, tablet and cell phone. They must leave all devices in a locked cabinet. If the devices are returned randomly, What is the probability that no student gets back a single of his/her devices? Write the answer as a product of rational numbers.

7. Consider the integers $\mathbb{Z}$. We define a relation $\preceq$ on $\mathbb{Z}$ by: $a \preceq b$ if one of the two conditions is satisfied: **i)** $a = b$ or **ii)** $|a| < |b|$.

   (a) Show that $\preceq$ is a partial ordering. (1p)
   (b) Show that $\preceq$ is not a total ordering. (1p)
   (c) Determine, if they exist, $\text{lub}\{5, -5\}$ and $\text{glb}\{5, -5\}$ (1p)



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Answers TATA82 Discrete Mathematics 3/6 2022

1) Show with Math.Ind. $\sum_{k=2}^{n} (k+1)^3 = \frac{n^4+6n^3+13n^2+12n-32}{4}$ $\forall n \geq 2$

We should: i) first verify the formula for

$n = 2$ $\quad LHS_2 \quad (2+1)^3 = 27 = \frac{2^4+6(2)^3+13(2)^2+12(2)-32}{4}$

ii) Assume that $\sum_{2}^{n} (k+1)^3 = \frac{n^4+6n^3+13n^2+12n-32}{4}$, for some $n \geq 2$

and we prove it for $n+1$:

$$LHS_{n+1} = \sum_{2}^{n+1} (k+1)^3 = (n+2)^3 + \sum_{2}^{n} (k+1)^3 \overset{\text{Assump}}{=}$$

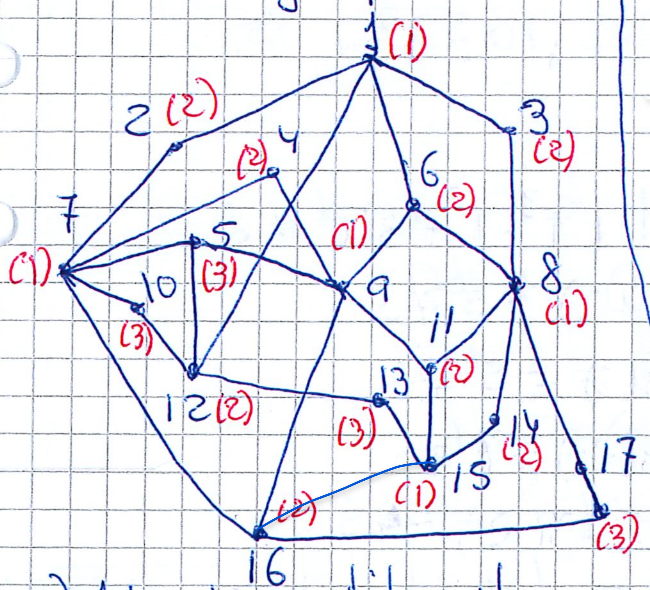
$$(n+2)^3 + \frac{n^4+6n^3+13n^2+12n-32}{4} =$$

$$\frac{4n^3+(6+18)n^2+(4+18+26)n+(1+6+13+12)+n^4+6n^3+13n^2+12n-32}{4}$$

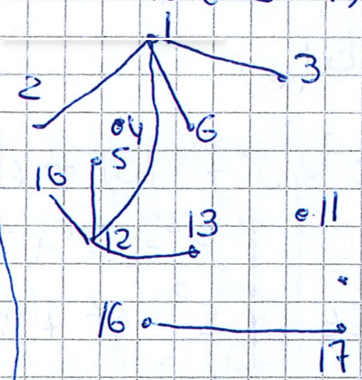
$$= \frac{(n+1)^4+6(n+1)^3+13(n+1)^2+12(n+1)-32}{4} = RHS_{n+1}.$$

As wanted

2) The graph

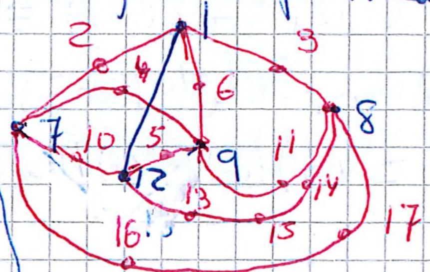


a) Is not Hamiltonian, take away the nodes 7, 9, 8 and 15 we get



with components As $\geq 4+1$; Non-Hamiltonian

b) Non-planar, there is the edge-deformation of K_5



c) Non-bipartite: it has the pentagon $8 \to 14 \to 15 \to 16 \to 17 \to 8$

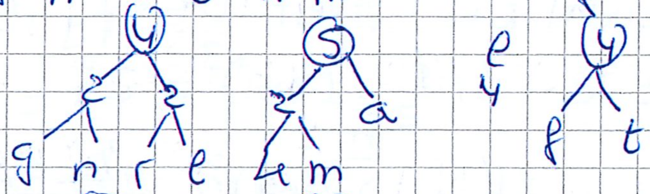
d) $\chi(G) \geq 3$ (non-bipartite) A coloring with 3 colors above $\chi(G) = 3$

3a) A cheapest Prefix Code for GENERAL KAFFEMATTE

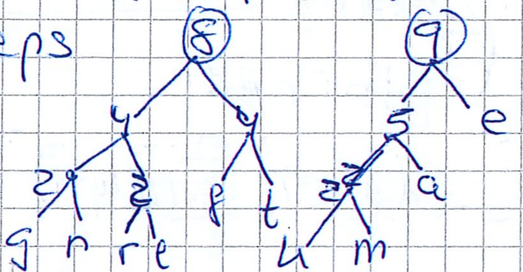
g	e	n	r	a	l	k	f	m	t
1	4	1	1	3	1	1	2	1	2

After 4 steps :

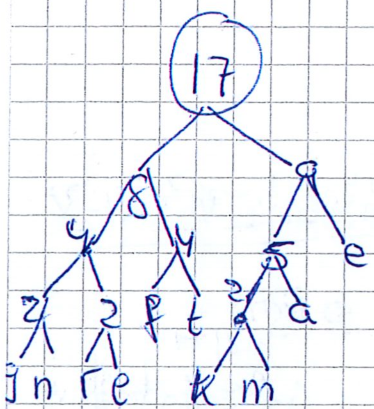
After other 2 steps



After other 2 steps



. The tree:



So
- g ↔ 0000
- e ↔ 11
- n ↔ 0001
- r ↔ 0010
- a ↔ 101
- l ↔ 0011
- k ↔ 1000
- f ↔ 010
- m ↔ 1001
- t ↔ 011

Encription 0000 11 0001 11 0010 101 0011 1000 101 010 010 11 1001 101 011 011 11

b) The binary representation of 10537_{10} is $10100100101001_{(2)}$

4a) We check $0^3-0^2+3(0)\not\equiv 6$ | $1^3-1^2+3(1)\not\equiv 6$

$2^3-2^2+3(2)\not\equiv 6$ | $3^3-3^2+3(3)\equiv 5\not\equiv 6$

$4^3-4^2+3(4)\equiv 5\not\equiv 6$ | $5^3-5^2+3(5)\equiv 5\not\equiv 6$

$(-5)^3-(-5)^2+3(-5)\equiv 0$ | $(-4)^3-4^2+3(-4)\equiv 7\not\equiv 6$

$(-3)^3-(-3)^2+3(-3)\equiv 10\not\equiv 6$ | $(-2)^3-(-2)^2+3(-2)\equiv 4\not\equiv 6$

$(-1)^3-(-1)^2+3(-1)\equiv 6 \mod 11$, Solution $\boxed{x=10}$

4b) We look for $\varphi(5005)=\varphi(5)\varphi(7)\varphi(11)\varphi(13)=$
$=(4)(6)(10)(12)=\underline{2880}$

5) Solve
$$\begin{cases} (1)\; a_{n+1} = 2a_n - b_n + (n+1) \\ (2)\; b_{n+1} = -2a_n + 3b_n + n^2 \end{cases} \qquad a_0 = 2,\ b_0 = 2 \quad (a_1 = 3,\ b_1 = 2)$$

from (1) $b_n = -a_{n+1} + 2a_n + (n+1)$

$$b_{n+1} = -a_{n+2} + 2a_{n+1} + (n+2)$$

Setting them in (2)

$$-a_{n+2} + 2a_{n+1} + (n+2) = -2a_n - 3a_{n+1} + 6a_n + 3n + 3 + n^2$$

$$a_{n+2} - 5a_{n+1} + 4a_n = -(n^2 + 2n + 1) = -(n+1)^2$$

i) $a_n^{(h)} = A_1(1)^n + A_2(4)^n$, since charac. eq. $r^2 - 5r + 4 = 0$

ii) $a_n^{(p)} = B_1 n^3 + B_2 n^2 + B_3 n$. Setting it in the eq.

$$B_1(n+2)^3 + B_2(n+2)^2 + B_3(n+2) - 5B_1(n+1)^3 - 5B_2(n+1)^2 - 5B_3(n+1)$$
$$+ 4B_1 n^3 + 4B_2 n^2 + 4B_3 n = -(n+1)^2 = -n^2 - 2n - 1$$

this gives $B_1 = \frac{1}{9}$, $B_2 = \frac{5}{18}$, $B_3 = \frac{19}{54}$

$$a_n = A_1 + A_2(4)^n + \frac{n^3}{9} + \frac{5n^2}{18} + \frac{19n}{54}.$$

The initial conditions give $\begin{cases} a_0 = 2 = A_1 + A_2 \\ a_1 = 3 = A_1 + 4A_2 + \frac{20}{27} \end{cases}$

This gives $A_1 = \frac{155}{81}$, $A_2 = \frac{7}{81}$

So $a_n = \frac{7}{81}(4)^n + \frac{n^3}{9} + \frac{5n^2}{18} + \frac{19n}{54} + \frac{155}{81}$

and $b_n = \frac{-14}{81}(4)^n + \frac{n^3}{9} - \frac{n^2}{18} + \frac{25n}{54} + \frac{176}{81}$

6) If no student gets a single of his/her devices then, with multiplication principle,

$$\left(\frac{d_6}{6!}\right)\left(\frac{d_6}{6!}\right)\left(\frac{d_6}{6!}\right)$$ since the probability of not getting a device is $\left\{\frac{\text{derangements of 6 symbols}}{\text{permutations of 6 symbols}}\right.$

Now $d_6 = \frac{6!}{2} - \frac{6!}{6} + \frac{6!}{24} - \frac{6!}{120} + \frac{6!}{720} = 265$

So, totally $\left(\frac{265}{720}\right)^3$

7) Consider the relation $\preccurlyeq$ on $\mathbb{Z}$ given by

$a \preccurlyeq b$ if $\begin{cases} \text{i) } a = b \\ \quad \text{or} \\ \text{ii) } |a| < |b| \end{cases}$

a) Show that $\preccurlyeq$ is a partial ordering: We prove

i) $\preccurlyeq$ reflexive since $\forall a \in \mathbb{Z}$, $a = a$ and $a \preccurlyeq a$

ii) $\preccurlyeq$ antisymmetric, since if $a \preccurlyeq b$ and $b \preccurlyeq a$ it cannot be that $|a| < |b|$ and $|b| < |a|$ so $a = b$

iii) $\preccurlyeq$ transitive, since if $a \preccurlyeq b$ and $b \preccurlyeq c$ it can happen

$$\begin{cases} a = b \text{ and } b = c \text{ so } a = c \rightarrow a \preccurlyeq c \\ |a| < |b| \text{ and } b = c \text{ so } |a| < |c| \rightarrow a \preccurlyeq c \\ a = b \text{ and } |b| < |c| \text{ so } |a| < |c| \rightarrow a \preccurlyeq c \\ |a| < |b| \text{ and } |b| < |c| \text{ so } |a| < |c| \rightarrow a \preccurlyeq c \end{cases}$$

b) $\preccurlyeq$ is not a total ordering since we cannot compare 5 and -5 because $\{$ i) $5 \neq -5$ but $|5| \not< |-5|$ and $|-5| \not< |5|$

c) lub $\{5, -5\}$ does not exist, since 6 and -6 are upper bounds

glb $\{5, -5\}$ does not exist, since 4 and -4 are lower bounds