Written Examination in Discrete Mathematics TATA82, TEN1, 2022–08–18, kl 08–13.

No calculator. Complete motivations required.

For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

1. Consider Fibonacci numbers, defined by $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}, n \geq 3$. Show using mathematical induction that $f_n \geq (\frac{1+\sqrt{5}}{2})^{n-2}$, for all $n \geq 3$.

2. (a) Consider a complete binary tree of height h where all leaves have level equal to the height of the tree. Determine the number of nodes and edges of the tree. (1p)

   (b) Can a connected planar graf with all the faces of degree 6 and all the nodes of degree 4 exist? (1p)

3. (a) In how many ways can one divide 25 dices in 5 boxes, if no box is empty? (1p)

   (b) In how many ways can one divide 25 people in 5 hotel rooms, with 5 people in each room? (1p)

   (c) In how many ways can one divide 25 people in 5 groups, with 5 in each group, where the order of the groups is irrelevant? (1p)

4. (a) Which negative integers u, v satisfy that $143u^3 + 187v^3 = -2640$. (1p)

   (b) Determine the integers between 1300 and 2600 that are congruent to $10^{123456789}$ mod 1287 (2p)

5. Solve the difference equation $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = (3n+1)(2)^n$, $a_0 = 2, a_1 = 14, a_2 = 42$.

6. Consider the functions $f : A \to \{0,1\}$, where $A = \{l = x_{10}x_9x_8x_7x_6x_5x_4x_3x_2x_1x_0\,;\ x_i \in \{0,1\}\}$ is the set of binary lists of length 11. Determine the number of such functions that satisfy at least one of the four following conditions: $f(0x_9x_8x_7x_6x_5x_4x_3x_2x_1x_0) = 0$, or $f(x_{10}x_9x_8x_7x_6x_5x_4x_3x_2x_10) = 0$, or $f(x_{10}x_9x_81x_6x_5x_4x_3x_2x_1x_0) = 0$, or $f(x_{10}x_9x_8x_7x_6x_5x_41x_2x_1x_0) = 0$.

7. Consider the set of integers $A = \{t \in \mathbb{Z}; 0 \leq t \leq 2047$. We define a relation $\mathcal{R}$ on A as follows: $t_1 \mathcal{R} t_2$ if their bit-representation have the same number of 0:s. Example $1024\mathcal{R}1$ since $1024 = 10000000000_{(2)}$ and $1 = 00000000001_{(2)}$.

   (a) Show that $\mathcal{R}$ is an equivalence relation. (1p)

   (b) How many equivalence classes are there?. (1p)

   (c) How many integers in A belong to the equivalence class of $919 = 01110010111_{(2)}$. (1p)

LINKÖPINGS UNIVERSITET
Matematiska institutionen
Milagros Izquierdo

Tentamen i Diskret Matematik, TATA82, TEN1, 2022–08–18, kl 08–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

---

1. Betrakta Fibonaccital som definieras av $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}, n \geq 3$. Visa med induktionsprincipen att $f_n \geq (\frac{1+\sqrt{5}}{2})^{n-2}$, för alla $n \geq 3$.

2. (a) Betrakta ett fulltändigt binärt träd av höjden h där alla löv sitter på höjden av trädet. Bestäm antal noder och kanter i trädet. (2p)

   (b) Kan det existera en sammanhängande planär graf där alla regioner har gradtal 6 och alla noder har gradtal 4? (1p)

3. (a) På hur många sätt kan man dela 25 tärningar i 5 lådor om ingen låda är tom? (1p)

   (b) På hur många sätt kan man dela 25 personer i 5 hotell rum, med 5 personer i varje? (1p)

   (c) På hur många sätt kan man dela 25 personer i 5 grupper, med 5 i varje, om ordningen i vilken grupper ligger är irrelevant? (1p)

4. (a) Vilka negativa heltal u, v uppfyller att $143u^3 + 187v^3 = -2640$. (1p)

   (b) Bestäm alla heltal mellan 1300 och 2600 som är kongruenta med $10^{123456789}$ mod 1287 (2p)

5. Lös den rekursiva ekvationen $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = (3n+1)(2)^n$, $a_0 = 2, a_1 = 14, a_2 = 42$.

6. Betrakta funktioner $f : A \to \{0,1\}$, där $A = \{l = x_{10}x_9x_8x_7x_6x_5x_4x_3x_2x_1x_0\ ;\ x_i \in \{0,1\}\}$ är mängden med binära listor av längd 11. Bestäm hur många funktioner ovan som uppfyller minst ett av följande villkor: $f(0x_9x_8x_7x_6x_5x_4x_3x_2x_1x_0) = 0$, eller $f(x_{10}x_9x_8x_7x_6x_5x_4x_3x_2x_10) = 0$, eller $f(x_{10}x_9x_81x_6x_5x_4x_3x_2x_1x_0) = 0$, eller $f(x_{10}x_9x_8x_7x_6x_5x_41x_2x_1x_0) = 0$?

7. Betrakta heltalen $A = \{t \in \mathbb{Z}; 0 \leq t \leq 2047$. Vi definierar en relation $\mathcal{R}$ på A genom: $t_1 \mathcal{R} t_2$ om deras bit-representationer har samma antal 0:or. Exempel $1024\mathcal{R}1$ eftersom $1024 = 10000000000_{(2)}$ och $1 = 00000000001_{(2)}$.

   (a) Visa att $\mathcal{R}$ är en ekvivalensrelation. (1p)

   (b) Bestäm antalet ekvivalensklasser. (1p)

   (c) Bestäm antalet heltal i ekvivalenklassen till $919 = 01110010111_{(2)}$. (1p)

Answers TATA82 Discrete Mathematics 18/8 2022

1) Show with Math Ind. that $\forall n \geq 3$ $f_n \geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}$, where f_n are Fibonacci numbers defined by
$f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$

To prove it with Math Induction we show

i) True for $n=3$: $f_3 = f_1 + f_2 = 2 \geq \left(\frac{1+\sqrt{5}}{2}\right)^{3-2}$

We also show that $f_2 = 1 \geq \left(\frac{1+\sqrt{5}}{2}\right)^0 = 1$

ii) Assume that $f_k \geq \left(\frac{1+\sqrt{5}}{2}\right)^{k-2}$ for k between 2 and some $n \geq 3$, and prove the inequality for $n+1$:

$$f_{n+1} \underset{f_{n+1}}{=} f_n + f_{n-1} \underset{\text{Assumpt. twice}}{\geq} \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-3}$$

$$\underset{n+1 \geq 4}{=} \left(\frac{1+\sqrt{5}}{2}\right)^{n-3}\left(\frac{1+\sqrt{5}}{2}+1\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n-3}\left(\frac{3+\sqrt{5}}{2}\right) =$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{n-3}\left(\frac{1+\sqrt{5}}{2}\right)^{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1-2}$$ as wanted

2) a) In the tree at each level there are 2^ℓ nodes, $0 \leq \ell \leq h$
$|V(T)| = \sum_{\ell=0}^{h} 2^\ell = 2^{h+1}-1$, as for any tree $|E(T)| = 2^{h+1}-1-1$

b) Such a graph must satisfy $\begin{cases} 4n = 2e \\ 6f = 2e \\ n - e + f = 2 \end{cases}$, so
$n = |V(G)|$, $e = |E(G)|$, $f = |\text{faces}|$

$\frac{e}{2} - e + \frac{e}{3} = 2$; $-\frac{1}{6}e = 2$. Impossible

Such a graph can not exist

3a) We want the different solutions of
$x_1 + x_2 + x_3 + x_4 + x_5 = 25$, $x_i \geq 1$, that is $\binom{24}{4}$

b) Divisions of 25 people in groups of 5, ordered groups $\binom{25}{5,5,5,5,5}$

c) If the groups are unordered: $\frac{1}{5!}\binom{25}{5,5,5,5,5}$

4a) Solve the Diophantine equation $u, v \le 0$

$143u^3 + 187v^3 = -2640$, or $143U + 187V = -2640$

where $U = u^3$, $V = v^3$, $U, V \le 0$

As $\gcd(143, 187) = 11$ and $11 \mid -2640$ we have solution. We know $11 = (4)13 + (-3)17$, $U_0 = 4$, $V_0 = -3$, also $-2640 = (-240)(11)$, $k = -240$

$$\begin{cases} U = (-240)(4) + 17n \le 0 \\ V = (-240)(-3) - 13n \le 0 \end{cases} n \in \mathbb{Z} \iff \begin{cases} 17n \le 960 \\ 13n \geq 720 \end{cases} \quad n = 56$$

$U = V = -8$, so $\underline{u = v = -2}$

b) We write $1287 = (9)(11)(13)$. As $\gcd(9,11) = \gcd(9,13) = \gcd(11,13) = 1$ we can use Ch. R. Th. We first determine

$$\begin{cases} 10^{123456789} \equiv b_1 \bmod 9 \\ 10^{123456789} \equiv b_2 \bmod 11 \\ 10^{123456789} \equiv b_3 \bmod 13 \end{cases} \iff \begin{cases} 1^{123456789} \equiv 1 \bmod 9 \\ (-1)^{123456789} \equiv -1 \bmod 11 \\ (1)^{10288065}(-3)^9 \equiv -27 \equiv -1 \bmod 13 \end{cases}$$

Use that $(10)^{12} \equiv 1 \bmod 13$

Now with Ch. R. Th

$10^{123456789} \equiv (1)(143)x_1 + (-1)(117)x_2 + (-1)(99)x_3 \bmod 1287$

where $143x_1 \equiv 1 \bmod 9$ | $117x_2 \equiv 1 \bmod 11$ | $99x_3 \equiv 1 \bmod 13$

$x_1 \equiv -1 \bmod 9 \equiv 8$ | $x_2 \equiv 8 \bmod 11$ | $x_3 \equiv 5 \bmod 13$

$10^{123456789} \equiv (143)(8) - (117)(8) - (99)(5) \equiv 2287 \bmod 1287$

5) Solve $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = (3n+1)(2)^n$

$a_0 = 2$, $a_1 = 14$, $a_2 = 42$

Charac. Eq $r^3 - 5r^2 + 8r - 4 = 0$, $r_1 = 1$, $m_1 = 1$, $r_2 = 2$, $m_2 = 2$

$a_n^{(h)} = A_1 + (A_2 n + A_3)(2)^n$

$\underline{a_n^{(p)} = (B_1 n^3 + B_2 n^2)(2)^n}$. Setting $a_n^{(p)}$ in the Eq

$$8B_1 n^3 + 8B_2 n^2 - 20B_1(n-1)^3 - 20B_2(n-1)^2 + 16B_1(n-2)^3 + 16B_2(n-2)^2$$
$$-4B_1(n-3)^3 - 4B_2(n-3)^2 = 24n+8$$
yields $B_1 = B_2 = 1 \quad a_n^{(p)} = (2)^n(n^3+n^2)$
$$a_n = A_1 + (2)^n(A_2 n + A_3 + n^3 + n^2)$$
Initial conditions $\begin{cases} a_0 = 2 = A_1 + A_3 \\ a_1 = 14 = A_1 + 2A_2 + 2A_3 + 4 \\ a_2 = 42 = A_1 + 8A_2 + 4A_3 + 48 \end{cases}$

yields $A_3 = 38$, $A_2 = -15$, $A_1 = -36$

$$\underline{a_n = -36 + (2)^n(n^3 + n^2 - 15n + 38)}$$

6) We consider $\mathcal{U} = \{ f : A \to \{0,1\} \}$ as $|A| = 2048$
$|\mathcal{U}| = 2^{2048}$. We consider
$A_1 = \{ f \in \mathcal{U}$ with $f(0x_9 \dots x_0) = 0 \}$
$A_2 = \{ f \in \mathcal{U}$ with $f(x_{10} \dots x_1 0) = 0 \}$
$A_3 = \{ f \in \mathcal{U}$ with $f(x_{10}x_9x_8 | x_6 \dots x_0) = 0 \}$ and
$A_4 = \{ f \in \mathcal{U}$ with $f(x_{10} \dots x_4 | x_2 x_1 x_0) = 0 \}$. Then
$A_1 \cap A_2 = \{ f \in \mathcal{U}$ with $f(0x_9 \dots x_1 0) = 0 \}$
$A_1 \cap A_3 = \{ f \in \mathcal{U}$ with $f(0x_9x_8 | x_6 \dots x_0) = 0 \}$
$A_1 \cap A_4 = \{ f \in \mathcal{U}$ with $f(0x_9 \dots x_4 | x_2 x_1 x_0) = 0 \}$
$A_2 \cap A_3 = \{ f \in \mathcal{U}$ with $f(x_{10}x_9x_8 | x_6 \dots x_1 0) = 0 \}$
$A_2 \cap A_4 = \{ f \in \mathcal{U}$ with $f(x_{10} \dots x_4 | x_2 x_1 0) = 0 \}$
$A_3 \cap A_4 = \{ f \in \mathcal{U}$ with $f(x_{10}x_9x_8 | x_6x_5x_4 | x_2x_1x_0) = 0 \}$
$A_1 \cap A_2 \cap A_3 = \{ f \in \mathcal{U}$ with $f(0x_9x_8 | x_6 \dots x_1 0) = 0 \}$
$A_1 \cap A_2 \cap A_4 = \{ f \in \mathcal{U}$ with $f(0x_9 \dots x_4 | x_2 x_1 0) = 0 \}$
$A_1 \cap A_3 \cap A_4 = \{ f \in \mathcal{U}$ with $f(0x_9x_8 | x_6x_5x_4 | x_2x_1x_0) = 0 \}$
$A_2 \cap A_3 \cap A_4 = \{ f \in \mathcal{U}$ with $f(x_{10}x_9x_8 | x_6x_5x_4 | x_2x_1 0) = 0 \}$
$A_1 \cap A_2 \cap A_3 \cap A_4 = \{ f \in \mathcal{U}$ with $f(0x_9x_8 | x_6x_5x_4 | x_2x_1 0) = 0 \}$

Now $|A_1| = |A_2| = |A_3| = |A_4| = 2^{1024}$

for instance $|A_1| = 2^{1024}$ because the 1024 lists beginning by 1 can go to 0 or 1

In the same way $|A_{i_1} \cap A_{i_2}| = 2^{512}$, besides $1 \leq i_1 < i_2 \leq 4$

$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = 2^{256}$ and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 2^{128}$

$1 \leq i_1 < i_2 < i_3 \leq 4$

We want $|A_1 \cup A_2 \cup A_3 \cup A_4| =$

$= 4|A_1| - 6|A_1 \cap A_2| + 4|A_1 \cap A_2 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= 4(2)^{1024} - 6(2)^{512} + 4(2)^{256} - (2)^{128}$

7) Consider $\mathcal{A} = \{ l = x_{10} \cdots x_0 \, ; \, x_i = 0 \text{ or } 1 \}$

Define $t_1 \, R \, t_2$ if the bit-representation of t_1 and t_2 have the same number of 0's

a) R is an equivalence relation since

i) R reflexive $\forall t_1 \in \mathcal{A}$, $t_1 R t_1$ since t_1 has only one bit-representation

ii) R symmetric: if $t_1 R t_2$ the bit-representations of t_1 and t_2 have the same no. of 0's, then $t_2 R t_1$

iii) R transitive: if $t_1 R t_2$ and $t_2 R t_3$ then $t_1 R t_3$ since the bit-representations of t_1, t_2, and t_3 have the same no. of 0's

b) There are 12 equivalence classes since a binary list of length 11 can have: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 or 11 0's

c) The equivalence class of 919 is formed by all the binary lists of length 11 and 4 0's, there are $\binom{11}{4} = (11)(10)(9)(8) = 7920$ lists