Written Examination in Discrete Mathematics TATA82, TEN1, 2021–08–19, kl 08–13.

No calculator. Complete motivations required.

For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

1. Show using mathematical induction that $\sum_{k=0}^{n} 2(-7)^k = \frac{1-(-7)^{n+1}}{4}$, för alla $n \geq 0$.

2. The network Mattek consists of eleven members with links as in the graph G below:

   (a) Member no. **i** wants to send a message through the network in such a way that the message passes from a member to a neighbour in the network. The message, beginning at member no. **i** must pass each member once and come back to member no. **i**. Is it possible? Reason the answer. (1p)

   (b) Is the graph planar? Reason the answer. (1p)

   (c) When members will communicate outside the network, neighbouring members need different channels. How many channels are needed for the whole network? Reason the answer. (1p)

3. Determine the probability that no group gets more than 15 participants when 60 participants are divided in 6 groups.

4. (a) Show that $(2021, 821)$ is a good public key to a RSA-system with total number $N = 2021$ and determine the associated private key. (1p)

   (b) Determine the integers n such that

$$\begin{cases} 4n - 8 = 6r \\ 7n + 9 = 5s \end{cases}$$

   with r, s integers. (2p)

5. Solve the difference equation $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = (3)^n$, $a_0 = 1, a_1 = 6, a_2 = 72$.

6. (a) In how many ways can one divide 10 students in 5, all different, double rooms? (1p)

   (b) In how many ways can one divide 10 students in 5 indistinguishable double rooms? (1p)

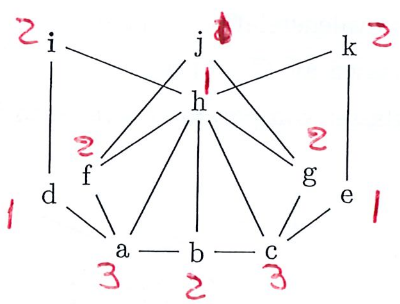
   (c) In how many ways can one divide 10 students in 4 hotel rooms with place for 1, 2, 3 and 4 students respectively? (1p)

7. Consider the points on the plane $A = \mathbb{R}^2$, and consider a fixed point C. We define a relation $\mathcal{R}$ on A as follows: $(x, y)\mathcal{R}(x', y')$ if the distance from (x, y) to C equals the distance from (x', y') to C.

   (a) Show that $\mathcal{R}$ is an equivalence relation. (1p)

   (b) Determine the equivalence class of C. (1p)

   (c) Show that the equivalence classes can be parametrized by non-negative real numbers r. (1p)



LINKÖPINGS UNIVERSITET
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Tentamen i Diskret Matematik, TATA82, TEN1, 2021–08–19, kl 08–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

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1. Visa med induktionsprincipen att $\sum_{k=0}^{n} 2(-7)^k = \frac{1-(-7)^{n+1}}{4}$, för alla $n \geq 0$.

2. Nätverket Mattek består av elva medlemmar med förbindelser som i grafen G nedan:

   (a) Medlem nr. **i** vill sprida ett meddelande genom att meddelandet överförs från en medlem till en granne i näverket på så sätt att varje medlem får meddelandet precis en gång och medlem nr. **i** får tillbaka meddelandet efter att alla har fått det. Är det möjligt? Motivera svaret. (1p)

   (b) Är grafen G planär? Motivera svaret. (1p)

   (c) När medlemmar vill kommunicera med omvärlden behöver grannarna använda olika kanaler, annars blir det olyckliga interferenser. Hur många kanaler behövs för hela nätverket? Motivera svaret. (1p)

3. Bestäm sannolikheten att ingen grupp har fler än 15 deltagare när man delar 60 deltagare i 6 grupper.

4. (a) Visa att $(2021, 821)$ är en bra offentlig nyckel till ett RSA-system med totalnummer $N = 2021$ och bestäm den associerade privata nyckeln. (1p)

   (b) Bestäm heltalen n sådana att

$$\begin{cases} 4n - 8 = 6r \\ 7n + 9 = 5s \end{cases}$$

   med r, s heltal. (2p)

5. Lös den rekursiva ekvationen $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = (3)^n$, $a_0 = 1, a_1 = 6, a_2 = 72$.

6. (a) På hur många sätt kan man dela 10 studenter i 5 namngivna hotellrum med plats för 2 studenter i varje? (1p)

   (b) På hur många sätt kan man dela 10 studenter i 5 lika (oskiljebara) hotellrum med plats för 2 studenter i varje? (1p)

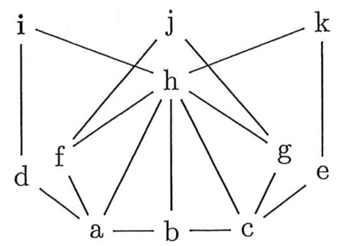
   (c) På hur många sätt kan man dela 10 studenter i 4 hotellrum med plats för 1, 2, 3 respektive 4 studenter? (1p)

7. Betrakta punkterna på planet $A = \mathbb{R}^2$, och betrakta en fixerad punkt C . Vi definierar en relation $\mathcal{R}$ på A genom: $(x, y)\mathcal{R}(x', y')$ om avståndet från (x, y) till C är lika med avståndet från (x', y') till C .

   (a) Visa att $\mathcal{R}$ är en ekvivalensrelation. (1p)

   (b) Bestäm ekvivalensklassen av C . (1p)

   (c) Visa att ekvivalensklasser kan parametriseras med icke-negativa reella tal r . (1p)



Answers TATA82 Discrete Maths 19/8 2021

1) Show by Math Induct. $\sum_{k=0}^{n} 2(-7)^k = \frac{1-(-7)^{n+1}}{4}, \ \forall n \geq 0$

i) For $n=0$ $VL_0 = 2(-7)^0 = 2 = \frac{1-(-7)}{4}$ True

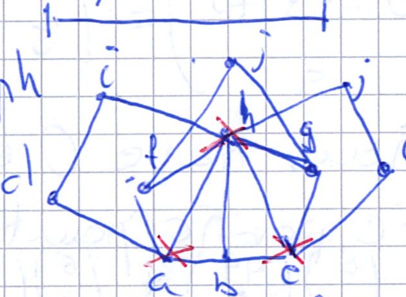
ii) Assume that the equality holds for some $n \geq 0$ and check that it is still true for $n+1$

$VL_{n+1} = \sum_{k=0}^{n+1} 2(-7)^k = \sum_{k=0}^{n} 2(-7)^k + 2(-7)^{n+1} \overset{\text{Assump.}}{=} \frac{1-(-7)^{n+1}}{4} +$

$+ 2(-7)^{n+1} = \frac{1}{4} + (-7)^{n+1}\left(-\frac{1}{4} + \frac{8}{4}\right) = \frac{1}{4} + (-7)^{n+1}\frac{(7)}{4}$

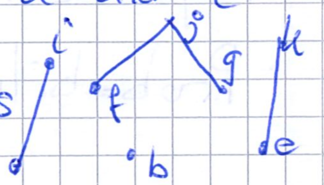
$= \frac{1}{4} - \frac{(-7)(-7)^{n+1}}{4} = \frac{1-(-7)^{n+2}}{4} = HL_{n+1}$. As required

2) a) the graph



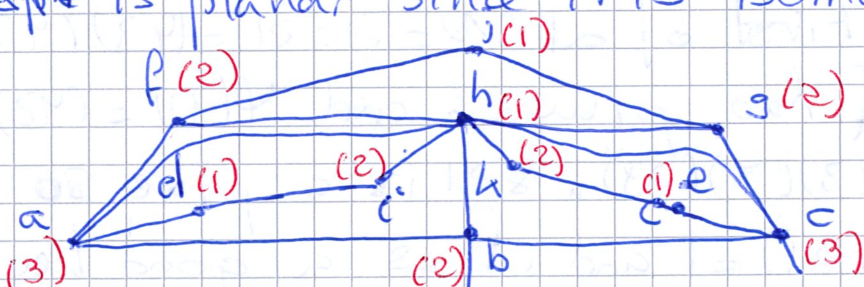
is Non-Hamiltonian since taking away the nodes "h", "a" and "c"

the resultant graph contains 4 components



and $4 \not\geq 3$. Member "i" can not do as wanted

b) Graph is planar since it is isomorphic to



c) The chromatic number, and the minimum no. of channels is $\underline{3}$, since the graph is not bipartite (it contains triangles) and there are colorings with 3 colors, see above ((1))

3) The probability that no group gets more then 15 participants when dividing 60 participants is $\dfrac{P_{\leq 15}}{\binom{60+5}{5}}$ where $P_{\leq 15}$ is the non-negative integer solutions of $x_1+x_2+x_3+x_4+x_5+x_6=60$
$0 \leq x_i \leq 15$, $1 \leq i \leq 6$

Using PIE $A_i = \{\text{solutions where } x_i \geq 16\}$ $1 \leq i \leq 6$

$P_{\leq 15} = |\overline{A_1 \cup \dots \cup A_6}| = \binom{60+5}{5} - |A_1 \cup \dots \cup A_6|$

$|A_i| = \binom{60-16+5}{5} = \binom{49}{5}$ $1 \leq i \leq 6$

$|A_{i_1} \cap A_{i_2}| = \binom{60-32+5}{5} = \binom{33}{5}$ $1 \leq i_1 < i_2 \leq 6$

$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \binom{60-48+5}{5} = \binom{17}{5}$, $1 \leq i_1 < i_2 < i_3 \leq 6$

All the other intersections have 0 elements since they require more then 60 participants so

$$\text{Probability} = \frac{\binom{65}{5} - \binom{6}{1}\binom{49}{5} + \binom{6}{2}\binom{33}{5} - \binom{6}{3}\binom{17}{5}}{\binom{65}{5}}$$

4

a) Show that $(N=2021, 821)$ is a good public key to a RSA-system and calculate the associated private key. First of all $N = 2021 = (43)(47)$ a product of two primes & and $\varphi(N) = (42)(46) = 1932$
$\varphi(2021) = (2)^2(3)(7)(23)$. 821 is a prime so $\gcd(1932, 821) = 1$ and it is a good key

We want $821a \equiv 1 \mod 1932$ or $1 = 821a + 1932y$
Using Euklides algorithm we get that

$$1 = 821(473) + 193(-201)$$

so $\underline{a = 473}$

4b) Solve, with integer solutions

$$\begin{cases} (1)\quad 4n - 6r = 8 \\ (2)\quad 7n + 5s = -9 \end{cases}$$

First of all $\gcd(4,6)=2 \mid 8$ and $\gcd(7,5)=1 \mid -9$

so $7(1) - 4(2)$ gives the Diophantine eq $20s + 42(-r) = 92$

$\gcd(20,42) = 2 \mid 92$ $\quad 10s + 21(-r) = 46$ with solution

$$\begin{cases} s = (-2)(46) + 21k \\ -r = (1)(46) - 10k \end{cases} \quad k \in \mathbb{Z}, \text{ since } 1 = \gcd(10,21) = (-2)10 + (1)21$$

$$\begin{cases} -r = 46 - 10k \\ r = -46 + 10k \end{cases}$$ and from (1) $4n = 8 + 6r$

$4n = 8 - 276 + 60k = -268 + 60k$

$8 + 15m = \boxed{n = -67 + 15k, \; k \in \mathbb{Z}}$

5) Solve $a_n - 8a_{n-1} + 21a_{n-2} + 18a_{n-3} = (3)^n$

$a_0 = 1$, $a_1 = 6$, $a_2 = 72$. The Charact. eq is

$r^3 - 8r^2 + 21r - 18 = 0$, with roots $r_1 = 2, m_1 = 1$, $r_{2,3} = 3, m_2 = 2$

so $a_n^{(h)} = A_1(2)^n + A_2(3)^n + A_3 n(3)^n$ and

$a_n^{(p)} = Bn^2(3)^n$, setting it i eq get

$$Bn^2(3)^n - 8B(n-1)^2(3)^{n-1} + 21B(n-2)^2(3)^{n-2} - 18B(n-3)^2(3)^{n-3} = (3)^n$$

$27Bn^2 - 72B(n-1)^2 + 63B(n-2)^2 - 18B(n-3)^2 = 27$, that gives

$B = \frac{27}{18} = \frac{3}{2}$ $\quad a_n^{(p)} = \frac{3n^2}{2}(3)^n$, $a_n = A_1(2)^n + A_2(3)^n + A_3 n(3)^n + \frac{3}{2}n^2(3)^n$

With I.C

$$\begin{cases} a_0 = 1 = A_1 + A_2 \\ a_1 = 6 = 2A_1 + 3A_2 + 3A_3 + \frac{9}{2} \\ a_2 = 72 = 4A_1 + 9A_2 + 18A_3 + 54 \end{cases} \qquad \begin{cases} 1 = A_1 + A_2 \\ 3/2 = 2A_1 + 3A_2 + 3A_3 \\ 18 = 4A_1 + 9A_2 + 18A_3 \end{cases}$$

This gives

$$\begin{cases} A_1 = 18 \\ A_2 = -17 \\ A_3 = 11/2 \end{cases}$$

$$\boxed{a_n = 18(2)^n + \left(\frac{3n^2 + 11n - 34}{2}\right)(3)^n}$$

6a) We divide the students in 5 different groups of 2 students : $\binom{10}{2,2,2,2,2} = \underline{113400}$

6b) Now, all the 5! ways of naming the 5 rooms is just $\underline{\text{one}}$. So we have $\frac{1}{5!}\binom{10}{2,2,2,2,2}$ $= \underline{945 \text{ ways}}$

6c) We divide the 10 students in 4 groups with 1, 2, 3 and 4 members respectively :

$$\binom{10}{1,2,3,4} = \underline{12600}$$

7 Consider a relation R on $\mathbb{R}^2$ as follows $(x,y) R (x',y')$ if $d((x,y), C) = d((x',y'), C)$

a) R is an equivalence relation since

i) $\underline{\forall} (x,y) \in \mathbb{R}^2 \quad d((x,y), C) = d((x,y), C)$

ii) If $(x,y) R (x',y')$ then $d((x,y), C) = d((x',y'), C)$ i.e $d((x',y'), C) = d((x,y), C)$, i.e $(x',y') R (x,y)$

and iii) If $(x,y) R (x',y')$ and $(x',y') R (\bar{x},\bar{y})$, then all $\underline{\text{three points}}$ have the same distance to C and also $d((x,y), C) = d((\bar{x},\bar{y}), C)$ i.e $(x,y) R (\bar{x},\bar{y})$

b) The equivalence class of C is

$\{(x,y) \in \mathbb{R}^2 ; d((x,y), C) = d(C, C) = 0\} = \underline{\{C\}}$

c) Consider $[P_0 = (x,y)] = \{P \in \mathbb{R}^2 ; d(P, C) = d(P_0, C)\}$

The distance $d_0 = d(P_0, C)$ determines the equivalence class $[P_0 = (x,y)]$ of P_0.

Each equivalence class is $\underline{\text{determined}}$, parametrized by the distance $(r \geq 0)$ of the elements in the class to C.