Written Examination in Discrete Mathematics TATA82, TEN1, 2021–06–03, kl 14–19.

No calculator. Complete motivations required.

For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

1. Consider Fibonacci numbers defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$. Show using mathematical induction that $-f_1 + f_2 + \cdots + (-1)^{2n-1} f_{2n-1} + (-1)^{2n} f_{2n} = \sum_{k=1}^{2n} (-1)^k f_k = f_{2n-1} - 1$, for all $n \geq 1$.

2. (a) Consider the graph G_1 below: Is it Hamiltonian? (1p)

   (b) Determine the chromatic number $\chi(G_1)$ of the graph G_1 below (1p)

   (c) A company trains 5 different candidates (P_i) to some of 5 possible jobs (J_j) as represented in the bipartite graph G_2 below. Show that there is an optimal matching where each candidate can get a job the candidate has been trained to. (1p)

3. How many functions $f : \{1, 2, 3, \dots n\} \to \{a, b, c\}$, $n \geq 3$ are there that assign c to exactly one positive integer strictly smaller than n?

4. Show that 30 is a divisor of $n^5 - n$ for all positive integers.

5. A sequence of negative integers $\{a_n\}$ is defined by the recursive equation $a_n^2 - a_{n-1}^2 = 4n^3 - 6n^2 + 8n - 3$, $n \geq 1$, $a_0 = -1$. Give a formula for a_n.

6. A company sends personal letters to each of its 100 employees randomly.

   (a) What is the probability that exactly 1 employee gets the right letter? (1p)

   (b) What is the probability that exactly 50 employees get the right letters? (1p)

   (c) What is the probability that exactly 99 employees get the right letters? (1p)

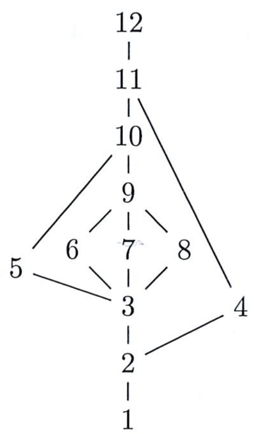
7. Companies and institutions use what is called *multilevel security check* to design the information flow. A company has 4 levels of security $L = \{0, 1, 2, 3\}$ and 4 categories of employees c_1, c_2, c_3, c_4 producing a set S of 16 (sub)sets of categories. For instance the set $\emptyset \in S$ corresponds to the employees that do not belong to any category, the set $\{c_1, c_2, c_3, c_4\} \in S$ corresponds to the employees that belong to some of the categories. Consider now the set $\mathcal{A} = \{(a, X);\ a \in L, X \in S\}$. We define a relation $\preceq$ on $\mathcal{A}$ by: $(a, X) \preceq (b, Y)$ if $a \leq b$ and $X \subset Y$. The company allows the information to flow from (a, X) to (b, Y) if $(a, X) \preceq (b, Y)$.

   (a) Show that $\preceq$ is a partial ordering. (1p)

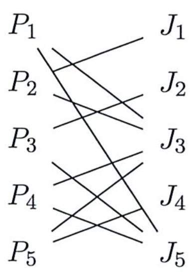
   (b) Show that $(\mathcal{A}, \preceq)$ is a lattice. (1p)

   (c) Determine the greatest and least elements in the lattice $(\mathcal{A}, \preceq)$ (1p)

G_1:



G_2:



LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA82, TEN1, 2021–06–03, kl 14–19.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

1. Betrakta Fibonaccital definierade av $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$. Visa med induktionsprincipen att $-f_1 + f_2 + \cdots + (-1)^{2n-1} f_{2n-1} + (-1)^{2n} f_{2n} = \sum_{k=1}^{2n} (-1)^k f_k = f_{2n-1} - 1$, för alla $n \geq 1$.

2. (a) Betrakta grafen G_1 nedan: Är den hamiltonsk? (1p)

   (b) Bestäm kromatiska talet $\chi(G_1)$ av grafen G_1 nedan (1p)

   (c) Ett företag tränar 5 kandidater (P_i) till några (J_j) av 5 olika jobb företaget erbjuder. Vilka jobb som kandidater tränats i finns representerat i bipartita grafen G_2 nedan. Visa att det finns en optimal matchning där varje kandidat kan få ett jobb som kandidaten har tränats i. (1p)

3. Hur många funktioner $f : \{1, 2, 3, \ldots n\} \to \{a, b, c\}$, $n \geq 3$ finns med villkoret att f ger värde c till exakt ett (1) positivt heltal strikt mindre än n?

4. Visa att 30 delar $n^5 - n$ för alla positiva heltal.

5. En följd av negativa heltal $\{a_n\}$ ges av rekursiva ekvationen $a_n^2 - a_{n-1}^2 = 4n^3 - 6n^2 + 8n - 3$, $n \geq 1$, $a_0 = -1$. Bestäm en formel för a_n.

6. Ett företag skickar personliga brev till var och en av dess 100 anställda slumpmässigt.

   (a) Vad är sannolikheten att exakt 1 anställd får det rätta brevet? (1p)

   (b) Vad är sannolikheten att exakt 50 anställda får de rätta breven? (1p)

   (c) Vad är sannolikheten att exakt 99 anställda får de rätta breven? (1p)

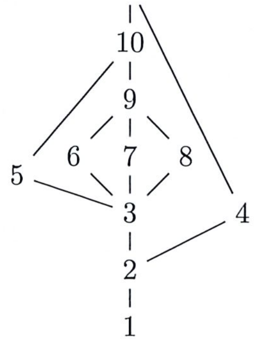
7. Företag och institutioner använder *multilevel security check* för att designa informationsflöde. Ett företag har 4 nivåer av säkerhet $L = \{0, 1, 2, 3\}$ och 4 kategorier av anställda c_1, c_2, c_3, c_4 som ger en mängd S med 16 (del)-mängder av kategorier. T.ex. $\emptyset \in S$ motsvarar de anställda som inte tillhör någon kategori, mängden $\{c_1, c_2, c_3, c_4\} \in S$ motsvarar de anställda som tillhör någon kategori. Betrakta mängden $\mathcal{A} = \{(a, X);\ a \in L, X \in S\}$. Vi definierar en relation $\preceq$ på $\mathcal{A}$: $(a, X) \preceq (b, Y)$ om $a \leq b$ och $X \subset Y$. Företaget tillåter att informationen flödar från (a, X) till (b, Y) om $(a, X) \preceq (b, Y)$.

   (a) Visa att $\preceq$ är en partialordning. (1p)

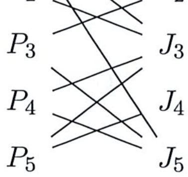
   (b) Visa att $(\mathcal{A}, \preceq)$ är en lattice. (1p)

   (c) Bestäm största och minsta element i latticen $(\mathcal{A}, \preceq)$ (1p)

G_1:



G_2:



Answers TATA 82 Discrete Maths 3/6 2021

1) Consider Fibonacci numbers: $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$

Show that $\sum_{k=1}^{2n} (-1)^k f_k = f_{2n-1} - 1 \quad \forall n \geq 1$

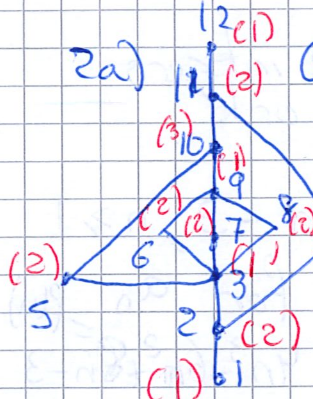
We prove it with Math Ind.:

i) Verify it is true for $n=1$ LHside $-f_1 + f_2 = 1-1 = 0 = f_1 - 1$ true

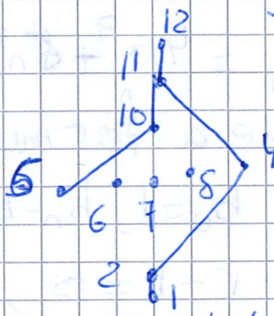
ii) Assume now that $\sum_{k=1}^{2n} (-1)^k f_k = f_{2n-1} - 1$ and prove the claim for $n+1$:

$$\text{LHside}_{n+1}: \sum_{k=1}^{2n+2} (-1)^k f_k = \sum_{k=1}^{2n} (-1) f_k - f_{2n+1} + f_{2n+2} \overset{\text{Assumpt}}{=} f_{2n-1} - 1 - f_{2n+1} + f_{2n+2}$$

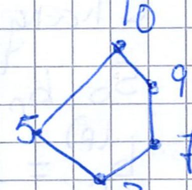
$$\underset{2n+2,\ 2n+1 \geq 3}{=} \cancel{f_{2n-1}} - 1 - \cancel{f_{2n-1}} - f_{2n} + \cancel{f_{2n+1}} + f_{2n} = f_{2n+1} - 1 \text{ as wanted}$$



2a) G_1 is non-Hamiltonian: for instance if we take away nodes 3 and 9 we get



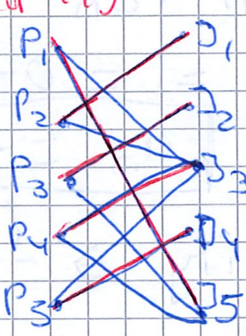
, which is a graph with 4 components and $4 > 3$



2b) G_1 is not bipartite; it contains the pentagon

So $\chi(G_1) \geq 3$, and $\chi(G_1) = 3$ with a coloring with 3 colors above in red (i)

2c) An optimal matching in



is $\begin{cases} e_1 = \{P_1, J_5\} \\ e_2 = \{P_2, J_1\} \\ e_3 = \{P_3, J_2\} \\ e_4 = \{P_4, J_3\} \\ e_5 = \{P_5, J_4\} \end{cases}$

3) Show that $30 \mid n^5 - n \quad \forall n \geq 1$

$n^5 - n = n(n-1)(n+1)(n^2+1)$. $\begin{cases} n^5 - n \equiv 0 \bmod 2 \text{ since } n \text{ or } n-1 \text{ are even} \\ n^5 - n \equiv 0 \bmod 3 \text{ since } n, (n-1) \text{ or } (n+1) \text{ is a multiple of } 3 \end{cases}$

We control that $n^5 - n \equiv 0 \mod 5$

n can be i) $n \equiv 0 \mod 5 \Rightarrow 0^5 - 0 \equiv 0 \mod 5$

ii) $n \equiv 1 \mod 5 \Rightarrow 1^5 - 1 \equiv 0 \mod 5$

iii) $n \equiv 2 \mod 5 \Rightarrow 2^5 - 2 \equiv 30 \equiv 0 \mod 5$

iv) $n \equiv 3 \mod 5 \Rightarrow 3^5 - 3 \equiv 340 \equiv 0 \mod 5$

v) $n \equiv 4 \equiv -1 \mod 5 \Rightarrow (-1)^5 - (-1) \equiv -1 + 1 \equiv 0 \mod 5$

as $\gcd(2,3) = \gcd(2,5) = \gcd(3,5)$

$n^5 - n \equiv 0 \mod 2$, $n^5 - n \equiv 0 \mod 3$, $n^5 - n \equiv 0 \mod 5$

$\Rightarrow n^5 - n \equiv 0 \ \mathrm{lcm}(2,3,5)$ i.e $n^5 - n \equiv 0 \mod 30$

3) Consider functions $f: \{1, 2, 3, \dots, n\} \to \{a, b, c\}$, $n \geq 3$

With the condition that exactly an integer i, $1 \leq i \leq n-1$ gets the value c. So the number of such functions is

$$\underbrace{\binom{n-1}{1}}_{\text{the integer going to } c} \underbrace{(3)}_{\text{the possible values for } n} \underbrace{(2)^{n-2}}_{\text{each of the other } n-2 \text{ integers can take values } a \text{ or } b} \boxed{= 3(n-1)2^{n-2}}$$

5) a_n satisfy $a_0 = -1$, $a_n^2 - a_{n-1}^2 = 4n^3 + 6n^2 + 8n - 3$, $n \geq 1$

a_n are negative integers. Give a formula for a_n

Change of variable $a_n^2 = b_n$ $\quad b_0 = 1$, $b_n - b_{n-1} = 4n^3 - 6n^2 + 8n - 3 = f(n)$, $n \geq 1$

So $b_n^{(h)} = A$, because char. eq. $r - 1 = 0$, $r_1 = 1$, $m_1 = 1$

$b_n^{(p)} = n(b_1 n^3 + b_2 n^2 + b_3 n + b_4)$ since $f(n) = p_3(n)(1)^n$

Setting $b_n^{(p)}$ in the eq we get $b_1 = 1$, $b_2 = 0$, $b_3 = 2$, $b_4 = 0$

$b_n^{(p)} = n^4 + 2n^2$, $b_n = n^4 + 2n^2 + A$

Init Cond $b_0 = 1 = 0 + A$ $\quad A = 1$, $b_n = n^4 + 2n^2 + 1 = (n^2+1)^2$

and $\boxed{a_n = -(n^2+1)}$, $n \geq 0$

6) a) Probability $\Rightarrow \dfrac{\binom{100}{1} d_{99}}{100!}$ $\left\{ \begin{array}{l} \text{1 chosen employee gets the right letter} \\ \text{derangement of the other 99} \end{array} \right.$

b) Probability of 50 $\Rightarrow \dfrac{\binom{100}{50} d_{50}}{100!}$ with the same reasoning

c) Probability of 99 $\Rightarrow \dfrac{\binom{100}{99} d_1}{100!} = 0$, remember $d_1 = 0$

7a) $\mathcal{A} = L \times S = \{(a, X) ; a \in L, X \in S\}$

$\preccurlyeq$ is defined by $(a, X) \preccurlyeq (b, Y)$ if $a \leq b, X \subseteq Y$

i) $\preccurlyeq$ reflexive since $\forall (a, X) \in \mathcal{A}$ $(a, X) \preccurlyeq (a, X)$

since $a \leq a$ and $X \subseteq X$

ii) $\preccurlyeq$ antisymmetric since if $(a, X) \preccurlyeq (b, Y)$ and

$(b, Y) \preccurlyeq (a, X)$ then $a \leq b$, $X \subseteq Y$ and

$b \leq a$, $Y \subseteq Y$ but then

$a = b$ $X = Y$

and $(a, X) = (b, Y)$

iii) $\preccurlyeq$ transitive since if $(a, X) \preccurlyeq (b, Y)$ and $(b, Y) \preccurlyeq (c, Z)$

then $a \leq b \leq c$ and $X \subseteq Y \subseteq Z$ then

$a \leq c$ and $X \subseteq Y$, i.e $(a, X) \preccurlyeq (c, Z)$

b) $(\mathcal{A}, \preccurlyeq)$ is a lattice: given $B = \{(a, X), (b, Y)\}$

$glb(B) = (\min\{a, b\}, X \cap Y)$

$lub(B) = (\max\{a, b\}, X \cup Y)$

c) greatest element is $(3, \{c_1, c_2, c_3, c_4\})$ since

for $a = 0, 1, 2, 3$ $a \leq 3$ and for any

subset $X \in S$, $X \subseteq \{c_1, c_2, c_3, c_4\}$

least element is $(0, \emptyset)$ since for

$a = 0, 1, 2, 3$ $0 \leq a$ and for any subset

$X \in S$ $\emptyset \subseteq X$.