Practice exam 1

Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Kurskod: TATA82
Modul: TEN1

Tentamen i TATA82 Diskret matematik

20YY-MM-DD kl X.00–(X+5).00

Inga hjälpmedel. Ej räknedosa.

På del A (uppgift 1–3) ska endast svar ges. De ska lämnas på ett gemensamt papper. *Varje uppgift på del A ger högst 1 poäng. Uppgifterna på del B (uppgift 4–8) ger högst 3 poäng per uppgift. Till dessa krävs fullständiga lösningar.*

Godkänt på alla tre kontrollskrivningar *KTR1–3 år 2024 adderar 1 bonuspoäng till totalpoängen. Markera detta genom att skriva "G" i rutan för uppgift 9 på skrivningsomslaget.*

För betyg 3/4/5 krävs 9/12/15 poäng totalt.

Lösningsförslag finns efter skrivtidens slut på kursens hemsida.

DEL A

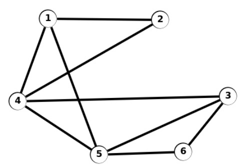
1. Hur många delmängder till $\{1,2,3\} \times \{1,2,3\}$ innehåller minst tre element?

2. Bestäm resten då $16^{23} \cdot 10^{65}$ divideras med 17.

3. Rita en icke-planär graf med sex hörn och kromatiskt tal 3.

DEL B

4. Betrakta grafen G i figuren till höger.



   (a) Ange en hamiltoncykel i G eller bevisa att ingen sådan finns.

   (b) Ange en eulerkrets i G eller bevisa att ingen sådan finns.

   (c) Beräkna det kromatiska polynomet $P(G, x)$.

5. Hur många "ord" (sekvenser av bokstäver) kan bildas av samtliga bokstäver i ordet PROTOTYP...

   (a) ...totalt?

   (b) ...om likadana bokstäver inte får stå intill varandra?

6. I ett RSA-kryptosystem har Bob den offentliga nyckeln $(55, 3)$. Alice krypterar ett meddelande och skickar kryptotexten "7" till Bob. Vad var klartexten som Alice krypterade?

7. Genom att stapla klossar på varandra byggs torn. Det finns tre olika sorters klossar: gula har höjd 1 cm, röda har höjd 2 cm och blå har också höjd 2 cm. Hur många olika torn av höjd n cm går att bygga, om n är ett positivt heltal? (Ignorera balanssvårigheter!)

8. Låt m och n vara heltal och $0 \leq m \leq n$. Visa att $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \binom{n}{m}2^{n-m}$.

(ENGLISH VERSION ON OPPOSITE PAGE)

Practice exam 1

Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Course code: TATA82
Module: TEN1

Examination in TATA82 Discrete mathematics

20YY-MM-DD at X.00–(X+5).00

No aid. No calculator.

In part A (problems 1–3), only answers shall be given. They are to be handed in on a single sheet of paper. *Each problem in part A is worth 1 point. The problems in part B (problems 4–8) are worth 3 points each. For them, complete solutions are required.*

Having passed all three digital tests *KTR1–3 in 2024 adds 1 bonus point to the total score. Indicate this by typing "G" in the box representing problem 9 on the exam cover.*

For grade 3/4/5 is required a total of 9/12/15 points.

After the exam, solutions are available from the course webpage.

PART A

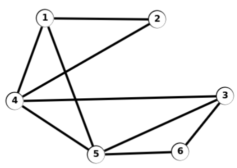
1. How many subsets of $\{1, 2, 3\} \times \{1, 2, 3\}$ contain at least three elements?

2. Compute the remainder when $16^{23} \cdot 10^{65}$ is divided by 17.

3. Draw a non-planar graph with six vertices and chromatic number 3.

PART B

4. Consider the graph G in the picture to the right.



   (a) Indicate a hamiltonian cycle in G or prove that there are none.

   (b) Indicate an eulerian circuit in G or prove that there are none.

   (c) Compute the chromatic polynomial $P(G, x)$.

5. How many "words" (sequences of letters) can be formed by using all letters in the word PROTOTYP...

   (a) ...in total?

   (b) ...if identical letters are not allowed to be adjacent?

6. In an RSA cipher, Bob has the public key $(55, 3)$. Alice encrypts a message and submits the ciphertext "7" to Bob. What was the plaintext that Alice encrypted?

7. By stacking bricks, towers are built. There are three different kinds of bricks: yellow have a height of 1 cm, red are 2 cm high, and blue are also 2 cm. How many different towers of height n cm can be built, if n is a positive integer? (Ignore balance issues!)

8. Let m and n be integers and $0 \leq m \leq n$. Show that $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \binom{n}{m} 2^{n-m}$.

(SVENSK VERSION PÅ OMSTÅENDE SIDA)

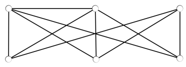
Solutions to practice exam 1

1. The set $\{1, 2, 3\} \times \{1, 2, 3\}$ contains $3 \cdot 3 = 9$ elements, hence has $2^9 = 512$ subsets. Of these, $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} = 46$ have less than three elements. **Answer:** 466.

2. Since 17 is a prime, $10^{16} \equiv 1 \pmod{17}$ by Fermat. Hence,

$$16^{23} \cdot 10^{65} \equiv (-1)^{23} \cdot 10 \cdot \left(10^{16}\right)^4 \equiv (-1) \cdot 10 \cdot 1^4 \equiv 7 \pmod{17}.$$

Answer: 7.

3. A graph obtained by adding one edge to the complete bipartite graph $K_{3,3}$ is non-planar since $K_{3,3}$ is. It has chromatic number at least 3 since it contains a 3-cycle. It has chromatic number at most 3 since it can be 3-coloured. **Answer:** 

4. (a) A hamiltonian cycle in G is $1, 5, 6, 3, 4, 2, 1$.
   (b) There is no eulerian circuit since there are vertices (1 and 3) of odd degree.
   (c) Suppose x is a positive integer. Any vertex colouring of G with at most x colours can by constructed by colouring the vertices in the following order: 1, 2, 4, 3, 5, 6. Then, when we are to colour a given vertex v, if two of its neighbours have already been coloured, they must have different colours because they are adjacent. Therefore, the number of available colours for v is x minus the number of already coloured neighbours of v. Thus, $P(G, x) = x(x-1)(x-2)(x-1)(x-2)(x-2)$.
   Answer: $x(x-1)^2(x-2)^3$.

5. (a) The number of ways to select two positions for the letters "O", two for the "P", two for the "T", and one each for "R" and "Y" out of eight available positions is the multinomial coefficient $\binom{8}{2,2,2,1,1} = 7! = 5040$. **Answer:** 5040.
   (b) Let X_O be the set of sequences that contain two adjacent "O", and define X_P and X_T similarly. By the principle of inclusion-exclusion, the number of sequences that contain adjacent identical letters is

$$|X_\mathrm{O}| + |X_\mathrm{P}| + |X_\mathrm{T}| - |X_\mathrm{O} \cap X_\mathrm{P}| - |X_\mathrm{O} \cap X_\mathrm{T}| - |X_\mathrm{P} \cap X_\mathrm{T}| + |X_\mathrm{O} \cap X_\mathrm{P} \cap X_\mathrm{T}|.$$

By considering pairs of identical letters such as "OO" as single symbols, we compute

$$|X_\mathrm{O}| = |X_\mathrm{P}| = |X_\mathrm{T}| = \binom{7}{2, 2, 1, 1, 1} = 1260,$$

$$|X_\mathrm{O} \cap X_\mathrm{P}| = |X_\mathrm{O} \cap X_\mathrm{T}| = |X_\mathrm{P} \cap X_\mathrm{T}| = \binom{6}{2, 1, 1, 1, 1} = 360,$$

and

$$|X_\mathrm{O} \cap X_\mathrm{P} \cap X_\mathrm{T}| = 5! = 120.$$

Summing up, and using the result from (a), the number of sequences that do not contain adjacent identical letters is

$$5040 - (3 \cdot 1260 - 3 \cdot 360 + 120) = 1220.$$

Answer: 1220.

6. Since $55 = 5 \cdot 11$, Bob's private key is the inverse of 3 modulo $4 \cdot 10 = 40$. Using Euler, or inspecting, we are led to notice $3 \cdot 27 \equiv 1 \pmod{40}$, meaning that the private key is 27. Computing modulo 55, we decrypt to find

$$7^{27} = 343^9 \equiv 13^9 = 13 \cdot 169^4 \equiv 13 \cdot 4^4 = 13 \cdot 256 \equiv 13 \cdot (-19) = -247 \equiv 28.$$

Answer: 28.

7. Let a_n be the number of different towers of height n. Suppose first that $n \geq 3$. Then there are a_{n-1} height n towers with a yellow brick on top since there is an arbitrary height $n-1$ tower below that brick. Similarly, there are a_{n-2} towers with a red top brick, and a_{n-2} towers with a blue top brick. Hence, $a_n = a_{n-1} + 2a_{n-2}$. Factoring the characteristic polynomial of this recurrence yields $x^2 - x - 2 = (x+1)(x-2)$. This shows that $a_n = c \cdot (-1)^n + d \cdot 2^n$ for some constants c and d. Using that $a_1 = 1$ and $a_2 = 3$ yields $c = \frac{1}{3}$ and $d = \frac{2}{3}$.

Answer: $\dfrac{(-1)^n + 2^{n+1}}{3}$.

8. The right hand side $\binom{n}{m}2^{n-m}$ is the number of ways to colour n different balls using three colours (yellow, red, blue) such that exactly m of the balls become yellow. This is because such a colouring is determined by a choice of m balls to colour yellow followed by a choice of a subset of the remaining $n - m$ balls to colour red (say).

   Now let us compute the number of such colourings in a different way. First choose those k balls that we shall colour yellow *or* red. For fixed k, we have $\binom{n}{k}$ choices. Then we choose which m out of those k balls to colour yellow; we have $\binom{k}{m}$ possibilities for that. Summing over all possible k yields the left hand side of the asserted identity. Hence it counts the same colourings as the right hand side does. $\square$