Practice exam 2

Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Kurskod: TATA82
Modul: TEN1

Tentamen i TATA82 Diskret matematik

20YY-MM-DD kl X.00–(X+5).00

Inga hjälpmedel. Ej räknedosa.

På del A (uppgift 1–3) ska endast svar ges. De ska lämnas på ett gemensamt papper. *Varje uppgift på del A ger högst 1 poäng. Uppgifterna på del B (uppgift 4–8) ger högst 3 poäng per uppgift. Till dessa krävs fullständiga lösningar.*

Godkänt på alla tre kontrollskrivningar *KTR1–3 år 2024 adderar 1 bonuspoäng till totalpoängen. Markera detta genom att skriva "G" i rutan för uppgift 9 på skrivningsomslaget.*

För betyg 3/4/5 krävs 9/12/15 poäng totalt.

Lösningsförslag finns efter skrivtidens slut på kursens hemsida.

DEL A

1. Hur många injektiva funktioner $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5, 6, 7\}$ uppfyller att $f(5)$ är udda?
2. Hur många icke-reflexiva relationer på $\{1, 2, 3, 4\}$ finns?
3. Finn två icke-isomorfa grafer som har samma sekvens av gradtal på hörnen.

DEL B

4. Vilket är det minsta $x \in \mathbb{N}$ som uppfyller både $x \equiv 4 \pmod{13}$ och $x \equiv 3 \pmod{17}$.
5. (a) Visa att bland 17 heltal finns det två vars differens är delbar med 16.
   (b) Antag att n är ett jämnt heltal. Visa att $n^3 + 6n^2 + 8n$ är delbart med 16.
   (c) Hur många element i $\{1, 2, \ldots, 4800\}$ är delbara med 16 men inte med 24?
6. Betrakta rekursionen $a_{n+2} = 3a_{n+1} - 2a_n - 6n^2 + 6n + 10$, $n \in \mathbb{N}$.
   (a) Visa med induktion att den lösning som uppfyller $a_0 = 0$ och $a_1 = 2$ är $a_n = 2n^3$.
   (b) Finn alla lösningar.
7. (a) Vad är definitionen av att en pomängd är ett *lattice*?
   (b) Antag att P är en icketom, ändlig pomängd i vilken varje par av element har en största undre begränsning. Visa att P är ett lattice om och endast om P har ett maximum (= största element).
8. Låt $n \geq 2$ vara ett heltal. Hur många $(n-2)$-reguljära delgrafer har den kompletta grafen K_n?

(ENGLISH VERSION ON OPPOSITE PAGE)

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Course code: TATA82
Module: TEN1

Examination in TATA82 Discrete mathematics

20YY-MM-DD at X.00–(X+5).00

No aid. No calculator.

In part A (problems 1–3), only answers shall be given. They are to be handed in on a single sheet of paper. *Each problem in part A is worth 1 point. The problems in part B (problems 4–8) are worth 3 points each. For them, complete solutions are required.*

Having passed all three digital tests *KTR1–3 in 2024 adds 1 bonus point to the total score. Indicate this by typing "G" in the box representing problem 9 on the exam cover.*

For grade 3/4/5 is required a total of 9/12/15 points.

After the exam, solutions are available from the course webpage.

PART A

1. How many injective functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5, 6, 7\}$ satisfy that $f(5)$ is odd?
2. How many non-reflexive relations on $\{1, 2, 3, 4\}$ exist?
3. Find two non-isomorphic graphs with the same vertex degree sequence.

PART B

4. Which is the smallest $x \in \mathbb{N}$ that satisfies both $x \equiv 4 \pmod{13}$ and $x \equiv 3 \pmod{17}$?
5. (a) Show that among 17 integers, there are two whose difference is divisible by 16.
   (b) Assume that n is an even integer. Show that $n^3 + 6n^2 + 8n$ is divisible by 16.
   (c) How many elements of $\{1, 2, \ldots, 4800\}$ are divisible by 16 but not by 24?
6. Consider the recurrence $a_{n+2} = 3a_{n+1} - 2a_n - 6n^2 + 6n + 10$, $n \in \mathbb{N}$.
   (a) Use induction to show that the solution which satisfies $a_0 = 0$ and $a_1 = 2$ is $a_n = 2n^3$.
   (b) Find all solutions.
7. (a) What is the definition of a poset being a *lattice*?
   (b) Suppose P is a nonempty, finite poset in which every pair of elements has a greatest lower bound. Show that P is a lattice if and only if P has a maximum (= greatest element).
8. Let $n \geq 2$ be an integer. How many $(n-2)$-regular subgraphs does the complete graph K_n have?

(SVENSK VERSION PÅ OMSTÅENDE SIDA)

Solutions to practice exam 2

1. Construct such a function by choosing $f(5)$ among 4 possibilities, then $f(4)$ among 6, $f(3)$ among 5, $f(2)$ among 4 and, finally, $f(1)$ among 3 possibilities.

   Answer: $4 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1440$.

2. Let $X = \{1, 2, 3, 4\}$. Since $X \times X$ has 16 elements, there are 2^{16} relations on X. Such a relation $\mathcal{R}$ is reflexive if and only if $\mathcal{R} \supseteq \{(1,1),(2,2),(3,3),(4,4)\}$. Hence there are 2^{16-4} reflexive relations. **Answer:** $2^{16} - 2^{12}$ $(= 61440)$.

3. There are many possibilities. One is to let the first graph be a 6-cycle and the second consist of two disjoint 3-cycles. Both have vertex degree sequence $2, 2, 2, 2, 2, 2$.

4. Since $-3 \cdot 17 + 4 \cdot 13 = 1$ (which can be discovered using Euclid's algorithm if one does not happen to observe it), -3 is the inverse of 17 modulo 13 and 4 is the inverse of 13 modulo 17. By the Chinese remainder theorem, the simultaneous solutions to the two congruences are $x = 4 \cdot (-3) \cdot 17 + 3 \cdot 4 \cdot 13 + 13 \cdot 17k = -48 + 221k$, $k \in \mathbb{Z}$. The smallest nonnegative solution is given by $k = 1$. **Answer:** $x = 173$.

5. (a) Since every integer is congruent modulo 16 to a nonnegative integer strictly smaller than 16, the pigeonhole principle guarantees that two of the integers are congruent modulo 16. Their difference is then divisible by 16. □

   (b) Factor the polynomial to find $n^3 + 6n^2 + 8n = n(n+2)(n+4)$. This is a product of three consecutive even integers. They are all divisible by 2, and at least one of them is divisible by 4. Hence, their product is divisible by $2 \cdot 2 \cdot 4 = 16$. □

   (c) The elements that are divisible by 16 are those of the form $16k$, $k \in \{1, \ldots, 300\}$. Since $\operatorname{lcm}(16, 24) = 48$, the elements that are both divisible by 16 and by 24 are those of the form 48ℓ, $\ell \in \{1, \ldots, 100\}$. **Answer:** 200.

6. (a) The base cases $n = 0$ and $n = 1$ hold since $2 \cdot 0^3 = 0 = a_0$ and $2 \cdot 1^3 = 2 = a_1$. Choose $n \in \mathbb{N}$ and assume in order to use induction that $a_k = 2k^3$ for all $0 \leq k \leq n+1$. We must show that $a_{n+2} = 2(n+2)^3$. By the induction assumption,

   $$\begin{aligned} a_{n+2} &= 3 \cdot 2(n+1)^3 - 2 \cdot 2n^3 - 6n^2 + 6n + 10 \\ &= 6n^3 + 18n^2 + 18n + 6 - 4n^3 - 6n^2 + 6n + 10 \\ &= 2n^3 + 12n^2 + 24n + 16 \\ &= 2(n+2)^3, \end{aligned}$$

   as desired. □

   (b) The characteristic polynomial is $x^2 - 3x + 2 = (x-1)(x-2)$. Hence, the homogeneous part of the solution is $a_n^{\text{hom}} = A \cdot 1^n + B \cdot 2^n = A + B \cdot 2^n$, for arbitrary A and B. By (a), a particular solution is $2n^3$. **Answer:** $a_n = A + B \cdot 2^n + 2n^3$, $A, B \in \mathbb{R}$.

7. (a) A poset P is a *lattice* if every pair of elements $a, b \in P$ has a greatest lower bound and a least upper bound.

   (b) First, assume P is a lattice. Since P is finite and nonempty, it has at least one maximal element. If it has more than one, two such elements cannot have a least upper bound, which contradicts that P is a lattice. This shows the "only if" part. For the "if" direction, assume that P has a maximum, $\hat{1}$. In order to show that P is a lattice, it remains to verify that two arbitrary elements $a, b \in P$ have a least upper bound. Since $\hat{1} \geq a, b$, they have *some* upper bound. Let x and y be arbitrary minimal elements among all upper bounds of a and b. Then x and y have a greatest lower bound z which, by definition of greatest lower bounds, satisfies $a, b \leq z$. Since x and y were minimal and $z \leq x, y$ it follows that $x = y = z$, so this element is the least upper bound of a and b. □

8. An $(n-2)$-regular simple graph must have at least $n-1$ vertices. Moreover, if it has exactly $n-1$ vertices, it is isomorphic to the complete graph K_{n-1}. The complete graph K_n has n different subgraphs that are isomorphic to K_{n-1} (construct one by deleting an arbitrary vertex (n choices) and all its incident edges from K_n).

   It remains to count all $(n-2)$-regular subgraphs of K_n that contain all n vertices. Such a subgraph G is precisely the complement $G = \overline{M}$ of a *perfect matching* M, i.e. a graph in which every vertex has exactly one neighbour. A perfect matching is constructed by partitioning the vertex set into pairs, a pair consisting of two vertices, where each is the other's unique neighbour. If n is odd, no such partition exists. If n is even, we construct one by first choosing the neighbour of 1 ($n-1$ possibilities), then the neighbour of the smallest remaining vertex ($n-3$ possibilities), and so on. In total there are $(n-1)(n-3)(n-5)\cdots 3 \cdot 1$ such partitions.

   Answer: n, if n is odd; $n + (n-1)(n-3)\cdots 3$, if n is even.