Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Kurskod: TATA82
Modul: TEN1

Tentamen i TATA82 Diskret matematik

2024-06-01 kl 14.00–19.00

Inga hjälpmedel. Ej räknedosa.

På del A (uppgift 1–3) ska endast svar ges. De ska lämnas på ett gemensamt papper. *Varje uppgift på del A ger högst 1 poäng. Uppgifterna på del B (uppgift 4–8) ger högst 3 poäng per uppgift. Till dessa krävs fullständiga lösningar.*

Godkänt på alla tre kontrollskrivningar *KTR1–3 år 2024 adderar 1 bonuspoäng till totalpoängen. Markera detta genom att skriva "G" i rutan för uppgift 9 på skrivningsomslaget.*

För betyg 3/4/5 krävs 9/12/15 poäng totalt.

Lösningsförslag finns efter skrivtidens slut på kursens hemsida.

DEL A

1. Hur många "ord" kan bildas av alla bokstäver i ordet CRUYFF om de båda F:en inte får stå intill varandra?
2. Hur många relationer på $\{1, 2, 3, 4, 5\}$ är både reflexiva och symmetriska?
3. Rita en graf G som har kromatiskt polynom $P(G, x) = x(x-1)^2(x-2)(x-3)$.

DEL B

4. Betrakta den diofantiska ekvationen $96x + 78y = c$, där c är en konstant.
   (a) Vilket är det minsta $c \in \mathbb{Z}_+$ för vilket ekvationen har heltalslösningar?
   (b) Bestäm alla heltalslösningar då $c = -30$.
5. Fyra personer ska fördela 18 identiska objekt mellan sig. På hur många sätt kan det göras om...
   (a) ...varje person ska få minst tre objekt?
   (b) ...varje person ska få minst ett och högst sju objekt?
6. (a) Låt $a, b \in \mathbb{Z}$, $n \in \mathbb{Z}_+$. Vad är definitionen av att a är en *invers* till b modulo n?
   (b) Visa att 23^{35} är en invers till 23 modulo 37.
   (c) Hur många positiva delare till $2^3 \cdot 3^4 \cdot 7^5 \cdot 13^6$ har inverser modulo 52?
7. Definiera a_n för $n \in \mathbb{N}$ genom rekursionen $a_n = 2a_{n-1} + a_{n-2} + 5a_{n-3}$ om $n \geq 3$ och begynnelsevärdena $a_0 = 1$, $a_1 = 2$, $a_2 = 8$. Visa att $2^n \leq a_n \leq 3^n$ för alla $n \in \mathbb{N}$.
8. Antag att G är en (enkel) planär graf i vilken varje hörn har gradtal minst 4. Visa att G innehåller minst en 3-cykel (= delgraf isomorf med K_3).

(ENGLISH VERSION ON OPPOSITE PAGE)

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Course code: TATA82
Module: TEN1

Examination in TATA82 Discrete mathematics

2024-06-01 at 14.00–19.00

No aid. No calculator.

In part A (problems 1–3), only answers shall be given. They are to be handed in on a single sheet of paper. *Each problem in part A is worth 1 point. The problems in part B (problems 4–8) are worth 3 points each. For them, complete solutions are required.*

Having passed all three digital tests *KTR1–3 in 2024 adds 1 bonus point to the total score. Indicate this by typing "G" in the box representing problem 9 on the exam cover.*

For grade 3/4/5 is required a total of 9/12/15 points.

After the exam, solutions are available from the course webpage.

PART A

1. How many "words" can be formed using all letters of the word CRUYFF if the two F's are not allowed to be adjacent?

2. How many relations on $\{1, 2, 3, 4, 5\}$ are both reflexive and symmetric?

3. Draw a graph G whose chromatic polynomial is $P(G, x) = x(x-1)^2(x-2)(x-3)$.

PART B

4. Consider the diophantine equation $96x + 78y = c$, where c is a constant.
   (a) Which is the smallest $c \in \mathbb{Z}_+$ for which the equation has integer solutions?
   (b) Find all integer solutions when $c = -30$.

5. Four people are to distribute 18 identical objects among themselves. In how many ways can it be done if...
   (a) ...each person must receive at least three objects?
   (b) ...each person must receive at least one and at most seven objects?

6. (a) Let $a, b \in \mathbb{Z}$, $n \in \mathbb{Z}_+$. What is the definition of a being an *inverse* of b modulo n?
   (b) Show that 23^{35} is an inverse of 23 modulo 37.
   (c) How many positive divisors of $2^3 \cdot 3^4 \cdot 7^5 \cdot 13^6$ have inverses modulo 52?

7. Define a_n for $n \in \mathbb{N}$ by the recurrence $a_n = 2a_{n-1} + a_{n-2} + 5a_{n-3}$ if $n \geq 3$ and the initial values $a_0 = 1$, $a_1 = 2$, $a_2 = 8$. Show that $2^n \leq a_n \leq 3^n$ for all $n \in \mathbb{N}$.

8. Suppose G is a (simple) planar graph in which every vertex has degree at least 4. Show that G contains at least one 3-cycle (= subgraph isomorphic to K_3).

(SVENSK VERSION PÅ OMSTÅENDE SIDA)

Solutions

1. The number of "words" altogether is the multinomial coefficient $\binom{6}{2,1,1,1,1} = 3 \cdot 5!$. The words that contain FF can be thought of as the permutations of the five-letter alphabet $\{\mathrm{C}, \mathrm{R}, \mathrm{U}, \mathrm{Y}, \mathrm{FF}\}$. There are $5!$ of those. **Answer:** $(3-1) \cdot 5! = 240$.

2. Let $X = \{1, 2, 3, 4, 5\}$. A relation $\mathcal{R} \subseteq X \times X$ is reflexive if and only if $(i, i) \in \mathcal{R}$ for all $i \in X$. It is symmetric if and only if it either contains both of (i, j) and (j, i) or else neither of them, for all $i, j \in X$, $i \neq j$. There are $\binom{5}{2} = 10$ pairs of such i and j. Hence, there are $2^{10} = 1024$ different reflexive and symmetric relations on X. **Answer:** 1024.

3. A graph obtained by adding one edge connecting one vertex of the complete graph K_4 to a fifth vertex has the desired chromatic polynomial. Indeed, we may construct a k-colouring of it by first colouring the vertices of K_4 using different colours in $k(k-1)(k-2)(k-3)$ ways, and finally colouring the fifth vertex in any colour except that of its neighbour: $(k-1)$ choices. **Answer:** 

4. (a) The equation has integer solutions if and only if $\gcd(96, 78)$ divides c. The smallest positive such c is $\gcd(96, 78) = \gcd(3 \cdot 2^5, 2 \cdot 3 \cdot 13) = 2 \cdot 3 = 6$. **Answer:** $c = 6$.

   (b) Divide by $\gcd(96, 78) = 6$ to obtain $16x + 13y = -5$. Since $16 \cdot (-4) + 13 \cdot 5 = 1$, which can be found using Euclid if necessary, $x = (-4) \cdot (-5) = 20$, $y = 5 \cdot (-5) = -25$ is one solution. Since $\gcd(16, 13) = 1$, this means that all solutions are given by $x = 20 + 13\ell$, $y = -25 - 16\ell$, $\ell \in \mathbb{Z}$. Using $k = \ell + 2$ looks a little nicer.
   Answer: $x = -6 + 13k$, $y = 7 - 16k$, $k \in \mathbb{Z}$.

5. First, let us observe that the number of nonnegative integer solutions ($\mathbb{N}$-*solutions*, for brevity) to $x_1 + x_2 + x_3 + x_4 = k$ is $\binom{k+3}{3}$ for any constant $k \in \mathbb{N}$, because such a solution can be represented as a sequence of k stars and $4 - 1 = 3$ bars. Denote this observation by $(*)$.

   Now to the problem. Say that person number i receives x_i objects. Then we want to count $\mathbb{N}$-solutions to the equation $x_1 + x_2 + x_3 + x_4 = 18$ subject to various extra restrictions on the variables:

   (a) Here, $x_i \geq 3$ for all i. Setting $y_i = x_i - 3$, we look for the number of $\mathbb{N}$-solutions to $y_1 + y_2 + y_3 + y_4 = 6$, which by $(*)$ is $\binom{6+3}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 3 \cdot 4 \cdot 7 = 84$. **Answer:** 84.

   (b) Now, $1 \leq x_i \leq 7$ for all i. With $y_i = x_i - 1$, this means that we want to count those of the $\binom{14+3}{3}$ $\mathbb{N}$-solutions to $y_1 + y_2 + y_3 + y_4 = 14$ that satisfy $y_i \leq 6$ for all i. Let A_i be the set of $\mathbb{N}$-solutions in which y_i violates this condition, i.e. in which $y_i \geq 7$. By symmetry, $|A_1| = |A_2| = |A_3| = |A_4|$. Setting $z_1 = y_1 - 7$, A_1 is the set of $\mathbb{N}$-solutions to $z_1 + y_2 + y_3 + y_4 = 7$. By $(*)$, there are $\binom{7+3}{3}$ of those.

   Similarly, all intersections $A_i \cap A_j$, $i \neq j$, have the same cardinality. More precisely, $|A_i \cap A_j| = 1$ if $i \neq j$ since if $y_i \geq 7$, $y_j \geq 7$ they must both be equal to 7 and the other two variables vanish since their sum is 14. This also implies that all intersections of more than two of the A_i are empty.

   The principle of inclusion-exclusion now shows that the number of $\mathbb{N}$-solutions that do not violate any of the conditions is

$$\begin{aligned}\binom{17}{3} - 4|A_1| + \binom{4}{2}|A_1 \cap A_2| &= \binom{17}{3} - 4 \cdot \binom{10}{3} + 6 \cdot 1 \\ &= \frac{17 \cdot 16 \cdot 15}{3 \cdot 2} - \frac{4 \cdot 10 \cdot 9 \cdot 8}{3 \cdot 2} + 6 \\ &= 17 \cdot 40 - 40 \cdot 12 + 6 \\ &= 40 \cdot (17 - 12) + 6 \\ &= 206.\end{aligned}$$

Answer: 206.

6. (a) That $ab \equiv 1 \pmod{n}$.
   (b) Since 37 is a prime and $37 \nmid 23$, Fermat implies $23^{35} \cdot 23 = 23^{36} \equiv 1 \pmod{37}$. $\square$
   (c) An integer d has an inverse modulo $52 = 2^2 \cdot 13$ if and only if $\gcd(d, 52) = 1$. The positive divisors d of $2^3 \cdot 3^4 \cdot 7^5 \cdot 13^6$ that satisfy $\gcd(d, 2^2 \cdot 13) = 1$ are those that have the form $d = 3^\alpha \cdot 7^\beta$ for $\alpha \in \{0,1,2,3,4\}$, $\beta \in \{0,1,2,3,4,5\}$. There are $5 \cdot 6$ such numbers. **Answer:** 30.

7. Let us use (strong) induction on n. The base cases $n \in \{0,1,2\}$ hold since $2^0 \le 1 \le 3^0$, $2^1 \le 2 \le 3^1$, and $2^2 \le 8 \le 3^2$. For the induction step, fix $n \ge 3$ and assume that $2^k \le a_k \le 3^k$ for all $0 \le k < n$. We shall prove that $2^n \le a_n \le 3^n$ by applying the assumption for $k = n-1$, $k = n-2$, and $k = n-3$. First, look at the lower bound:

$$\begin{aligned}a_n &= 2a_{n-1} + a_{n-2} + 5a_{n-3} \\ &\ge 2 \cdot 2^{n-1} + 2^{n-2} + 5 \cdot 2^{n-3} \\ &\ge 2 \cdot 2^{n-1} \\ &= 2^n,\end{aligned}$$

as desired. Now for the upper bound:

$$\begin{aligned}a_n &= 2a_{n-1} + a_{n-2} + 5a_{n-3} \\ &\le 2 \cdot 3^{n-1} + 3^{n-2} + 5 \cdot 3^{n-3} \\ &= 3^{n-3}(18 + 3 + 5) \\ &\le 3^{n-3} \cdot 27 \\ &= 3^n,\end{aligned}$$

concluding the proof. $\square$

8. Let c be the number of connected components, v the number of vertices, e the number of edges, and f the number of faces (regions) of a plane embedding of G. Suppose, in order to obtain a contradiction, that G does not contain a 3-cycle. Then, every face of the embedding is incident to at least 4 edges. Hence, the number of edge-face incidences is at least $4f$. On the other hand, since every edge is incident to at most 2 faces, the number of such incidences is at most $2e$. Thus, $4f \le 2e$. Finally, by the handshake lemma, $4v \le 2e$. Combining these two inequalities with Euler's formula yields

$$1 + c = v - e + f \le \frac{2e}{4} - e + \frac{2e}{4} = 0,$$

which is absurd. This is the desired contradiction. $\square$