Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Kurskod: TATA82
Modul: TEN1

Tentamen i TATA82 Diskret matematik

2025-01-08 kl 8.00–13.00

Inga hjälpmedel. Ej räknedosa.

På del A (uppgift 1–3) ska endast svar ges. De ska lämnas på ett gemensamt papper. *Varje uppgift på del A ger högst 1 poäng. Uppgifterna på del B (uppgift 4–8) ger högst 3 poäng per uppgift. Till dessa krävs fullständiga lösningar.*

Godkänt på alla tre kontrollskrivningar *KTR1–3 år 2024 adderar 1 bonuspoäng till totalpoängen. Markera detta genom att skriva "G" i rutan för uppgift 9 på skrivningsomslaget.*

För betyg 3/4/5 krävs 9/12/15 poäng totalt.

Lösningsförslag finns efter skrivtidens slut på kursens hemsida.

DEL A

1. Hur många "ord" (bokstavsföljder) kan bildas av samtliga bokstäver i ordet DONKEY utan att ordet END förekommer som sammanhängande delföljd?

2. Om 19 har en invers modulo 26, beräkna inversen. Om 19 inte har en invers modulo 26, berätta det.

3. Rita en enkel graf som är bipartit och reguljär men inte eulersk.

DEL B

4. Nio olika objekt fördelas i tre olika lådor. På hur många sätt kan det göras, om...

   (a) ...varje låda ska innehålla tre objekt?

   (b) ...första lådan ska innehålla precis två objekt?

   (c) ...precis en låda ska vara tom?

5. Definiera a_n för $n \in \mathbb{Z}_+$ genom att låta $a_1 = 2$, $a_2 = 3$ och $a_n = a_{n-1} + a_{n-2}$ om $n \geq 3$.

   (a) Visa att för alla $n \in \mathbb{Z}_+$ är a_n antalet "ord" med n bokstäver ur alfabetet $\{\mathrm{A}, \mathrm{B}\}$ i vilka två A aldrig står intill varandra.

   (b) Visa att $a_{n+1}a_n = 2 + \sum_{j=1}^{n} a_j^2$ för alla $n \in \mathbb{Z}_+$.

6. (a) Låt G vara en enkel graf. Hur definieras det *kromatiska talet* $\chi(G)$?

   (b) Antag att G är en enkel graf på 10 hörn med $\chi(G) = 3$. Visa att komplementgrafen $\overline{G}$ har en delgraf som är isomorf med den kompletta grafen K_4.

7. Bestäm alla heltalslösningar till kongruensekvationen $x^3 - 2x + 11 \equiv 0 \pmod{105}$.

8. Antag att $S \subseteq \{1, 2, \ldots, 50\}$ uppfyller $|S| = 9$. Visa att det finns två olika delmängder $T \subseteq S$ och $U \subseteq S$ sådana att summan av talen i T är lika med summan av talen i U.

(ENGLISH VERSION ON OPPOSITE PAGE)

Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Course code: TATA82
Module: TEN1

Examination in TATA82 Discrete mathematics

2025-01-08 at 8.00–13.00

No aid. No calculator.

In part A (problems 1–3), only answers shall be given. They are to be handed in on a single sheet of paper. *Each problem in part A is worth 1 point. The problems in part B (problems 4–8) are worth 3 points each. For them, complete solutions are required.*

Having passed all three digital tests *KTR1–3 in 2024 adds 1 bonus point to the total score. Indicate this by typing "G" in the box representing problem 9 on the exam cover.*

For grade 3/4/5 is required a total of 9/12/15 points.

After the exam, solutions are available from the course webpage.

PART A

1. How many "words" (letter sequences) can be formed using all letters of the word DONKEY without the word END appearing as a consecutive subsequence?
2. If 19 has an inverse modulo 26, compute the inverse. If 19 does not have an inverse modulo 26, say so.
3. Draw a simple graph which is bipartite and regular but not eulerian.

PART B

4. Nine different objects are distributed in three different boxes. In how many ways can it be done, if. . .
   (a) . . . each box must contain three objects?
   (b) . . . the first box must contain exactly two objects?
   (c) . . . exactly one box must be empty?
5. Define a_n for $n \in \mathbb{Z}_+$ by letting $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2}$ if $n \geq 3$.
   (a) Show that for all $n \in \mathbb{Z}_+$, a_n is the number of "words" on n letters from the alphabet $\{\mathrm{A}, \mathrm{B}\}$ in which two A are never adjacent.
   (b) Show that $a_{n+1}a_n = 2 + \sum_{j=1}^{n} a_j^2$ for all $n \in \mathbb{Z}_+$.
6. (a) Let G be a simple graph. What is the definition of the *chromatic number* $\chi(G)$?
   (b) Suppose G is a simple graph on 10 vertices with $\chi(G) = 3$. Show that the complement graph $\overline{G}$ has a subgraph which is isomorphic to the complete graph K_4.
7. Find all integer solutions to the congruence equation $x^3 - 2x + 11 \equiv 0 \pmod{105}$.
8. Suppose $S \subseteq \{1, 2, \ldots, 50\}$ satisfies $|S| = 9$. Show that there are two different subsets $T \subseteq S$ and $U \subseteq S$ such that the sum of the numbers in T equals the sum of the numbers in U.

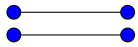
(SVENSK VERSION PÅ OMSTÅENDE SIDA)

Solutions

1. The number of permutations of $\{\mathrm{D, O, N, K, E, Y}\}$ minus the number of permutations of $\{\mathrm{END, O, K, Y}\}$ is $6! - 4! = 720 - 24$. **Answer:** 696.

2. Since $11 \cdot 19 = 209 = 26 \cdot 8 + 1 \equiv 1 \pmod{26}$, 11 is the inverse. Euclid's algorithm can be used to discover this. **Answer:** 11.

3. A graph with two disjoint edges on four vertices is one of many possibilities.

   Answer: 

4. (a) This is precisely the multinomial coefficient $\binom{9}{3,3,3} = \frac{9!}{3!3!3!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$. **Answer:** 1680.

   (b) Choose the objects that go into the first box in $\binom{9}{2} = 36$ ways. Then choose any subset of the remaining 7 objects for the second box in $2^7 = 128$ ways. This determines the content of the third box. **Answer:** $36 \cdot 128 = 4608$.

   (c) Choose which box to keep empty in 3 ways. Then choose any set of objects for the first of the remaining two boxes, except that this set must not be empty and must not contain all objects (for then the final box would become empty). Hence, there are $2^9 - 2 = 510$ such choices. **Answer:** $3 \cdot 510 = 1530$.

5. (a) Let b_n denote the number of n-letter words of the form specified in the problem; call such a word *allowed*. We show that $b_n = a_n$ for all $n \in \mathbb{Z}_+$ by (strong) induction. The allowed words A, B, AB, BA, and BB provide the base cases $b_1 = 2 = a_1$ and $b_2 = 3 = a_2$. For the induction step, fix $n \geq 3$ and assume $a_m = b_m$ whenever $m < n$. An allowed n-letter word either consists of an allowed $(n-1)$-letter word with B appended, or else of an allowed $(n-2)$-letter word with BA appended. Hence, $b_n = b_{n-1} + b_{n-2} = a_{n-1} + a_{n-2} = a_n$, as desired. □

   (b) We induct on n. When $n = 1$, the assertion is $a_2 a_1 = 2 + a_1^2$, which is true. Assume, for the induction step, that the assertion holds for some fixed $n \in \mathbb{Z}_+$. We verify that this implies $a_{n+2}a_{n+1} = 2 + \sum_{j=1}^{n+1} a_j^2$ by computing

$$a_{n+2}a_{n+1} = (a_{n+1} + a_n)a_{n+1} = a_{n+1}^2 + a_{n+1}a_n = a_{n+1}^2 + 2 + \sum_{j=1}^{n} a_j^2 = 2 + \sum_{j=1}^{n+1} a_j^2. \quad \square$$

6. (a) The minimal $k \in \mathbb{N}$ for which G has a proper k-colouring.

   (b) Since $\chi(G) = 3$, there is a proper 3-colouring of G. Choose such a colouring. Since $10/3 > 3$, there are four vertices that receive the same colour. Hence, no two of them are connected by an edge of G. Therefore, any two of them are connected by an edge of $\overline{G}$, meaning that they are the vertices of a subgraph of $\overline{G}$ isomorphic to K_4. □

7. Let $p(x) = x^3 - 2x + 11$. We have the prime factorization $105 = 3 \cdot 5 \cdot 7$. Hence, $p(x)$ is divisible by 105 if and only if it is congruent to 0 modulo 3, 5, and 7. We compute all possible values of $p(x)$ modulo those moduli:

$$\begin{aligned}
p(-1) &= 12 \equiv 0 \pmod 3, \\
p(0) &= 11 \not\equiv 0 \pmod 3, \\
p(1) &= 10 \not\equiv 0 \pmod 3, \\
p(-2) &= 7 \not\equiv 0 \pmod 5, \\
p(-1) &= 12 \not\equiv 0 \pmod 5, \\
p(0) &= 11 \not\equiv 0 \pmod 5, \\
p(1) &= 10 \equiv 0 \pmod 5, \\
p(2) &= 15 \equiv 0 \pmod 5, \\
p(-3) &= -10 \not\equiv 0 \pmod 7, \\
p(-2) &= 7 \equiv 0 \pmod 7, \\
p(-1) &= 12 \not\equiv 0 \pmod 7, \\
p(0) &= 11 \not\equiv 0 \pmod 7, \\
p(1) &= 10 \not\equiv 0 \pmod 7, \\
p(2) &= 15 \not\equiv 0 \pmod 7, \\
p(3) &= 32 \not\equiv 0 \pmod 7.
\end{aligned}$$

Hence, we look for the integers x that satisfy

$$\begin{cases} x \equiv -1 \pmod 3, \\ x \equiv 1 \text{ or } 2 \pmod 5, \\ x \equiv -2 \pmod 7. \end{cases}$$

By the Chinese remainder theorem, the solutions modulo 105 to this system are given by

$$\begin{aligned}
x &\equiv -1 \cdot (-1) \cdot 5 \cdot 7 + (1 \text{ or } 2) \cdot 1 \cdot 3 \cdot 7 + (-2) \cdot 1 \cdot 3 \cdot 5 \\
&= 35 + (21 \text{ or } 42) - 30 \\
&= 26 \text{ or } 47.
\end{aligned}$$

Answer: $x = 26 + 105k,\ k \in \mathbb{Z}$ or $x = 47 + 105k,\ k \in \mathbb{Z}$.

8. There are $2^9 = 512$ different subsets of S. Since $50 + 49 + \cdots + 42 < 9 \cdot 50$, the sum of the numbers in such a subset is in $\{0, 1, \ldots, 449\}$. In particular, there are less than 512 different values that the sum of the numbers in such a subset can attain. By the pigeonhole principle, there are two different subsets with the same value. $\square$