Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Kurskod: TATA82
Modul: TEN1

Tentamen i TATA82 Diskret matematik

2025-06-05 kl 14.00–19.00

Inga hjälpmedel. Ej räknedosa.

På del A (uppgift 1–3) ska endast svar ges. De ska lämnas på ett gemensamt papper. *Varje uppgift på del A ger högst 1 poäng. Uppgifterna på del B (uppgift 4–8) ger högst 3 poäng per uppgift. Till dessa krävs fullständiga lösningar.*

Godkänt på alla tre kontrollskrivningar *KTR1–3 år 2024 eller 2025 adderar 1 bonuspoäng till totalpoängen. Markera detta genom att skriva ”G” i rutan för uppgift 9 på skrivningsomslaget.*

För betyg 3/4/5 krävs 9/12/15 poäng totalt.

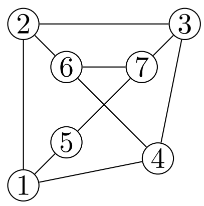
Lösningsförslag finns efter skrivtidens slut på kursens hemsida.

DEL A

1. Hur många funktioner $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$ är inte injektiva?
   Påminnelse: f kallas injektiv om $a \neq b \Rightarrow f(a) \neq f(b)$.
2. Partialordningen $\preceq$ på mängden av positiva delare till 24 definieras genom att låta $a \preceq b$ betyda att a delar b. Rita Hassediagrammet för denna pomängd.
3. Rita en enkel graf på fem hörn som har exakt tre uppspännande träd.

DEL B

4. Låt G vara grafen som är ritad i figuren till höger.



   (a) Avgör om G är hamiltonsk.

   (b) Avgör om G är bipartit.

   (c) Avgör om G är planär.

5. Lös rekursionsekvationen $a_{n+2} = 2a_{n+1} + 3a_n + 8 \cdot (-1)^n$, $n \in \mathbb{N}$, med begynnelsevärdena $a_0 = 4$ och $a_1 = -2$.
6. (a) Finn alla $x \in \mathbb{Z}$ som löser $\begin{cases} x \equiv 1 \pmod{3}, \\ x \equiv 2 \pmod{8}, \\ x \equiv 3 \pmod{11}. \end{cases}$

   (b) Visa att det inte finns något $x \in \mathbb{Z}$ som löser $\begin{cases} x \equiv 1 \pmod{3}, \\ x \equiv 2 \pmod{9}, \\ x \equiv 3 \pmod{11}. \end{cases}$
7. (a) Visa att $8^m \equiv 8 \pmod{28}$ för alla $m \in \mathbb{Z}_+$.

   (b) Finn alla heltal $n \geq 2$ som uppfyller att $8^m \equiv 8 \pmod{n}$ för alla $m \in \mathbb{Z}_+$.
8. Hur många olika ”ord” (bokstavsföljder) kan bildas av samtliga bokstäver i COUSCOUS om likadana bokstäver inte får stå intill varandra?

(ENGLISH VERSION ON OPPOSITE PAGE)

Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Course code: TATA82
Module: TEN1

Examination in TATA82 Discrete mathematics

2025-06-05 at 14.00–19.00

No aid. No calculator.

In part A (problems 1–3), only answers shall be given. They are to be handed in on a single sheet of paper. *Each problem in part A is worth 1 point. The problems in part B (problems 4–8) are worth 3 points each. For them, complete solutions are required.*

Having passed all three digital tests *KTR1–3 in 2024 or 2025 adds 1 bonus point to the total score. Indicate this by writing "G" in the box representing problem 9 on the exam cover.*

For grade 3/4/5 is required a total of 9/12/15 points.

After the exam, solutions are available from the course webpage.

PART A

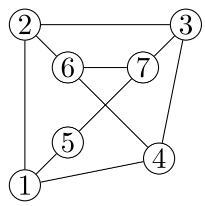
1. How many functions $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$ are not injective?
   Reminder: f is called injective if $a \neq b \Rightarrow f(a) \neq f(b)$.

2. The partial order $\preceq$ on the set of positive divisors of 24 is defined by letting $a \preceq b$ mean that a divides b. Draw the Hasse diagram of this poset.

3. Draw a simple graph on five vertices which has exactly three spanning trees.

PART B

4. Let G be the graph which is drawn in the figure to the right.



   (a) Decide whether G is hamiltonian.

   (b) Decide whether G is bipartite.

   (c) Decide whether G is planar.

5. Solve the recurrence equation $a_{n+2} = 2a_{n+1} + 3a_n + 8 \cdot (-1)^n$, $n \in \mathbb{N}$, with the initial values $a_0 = 4$ and $a_1 = -2$.

6. (a) Find all $x \in \mathbb{Z}$ that solve $\begin{cases} x \equiv 1 \pmod{3}, \\ x \equiv 2 \pmod{8}, \\ x \equiv 3 \pmod{11}. \end{cases}$

   (b) Show that no $x \in \mathbb{Z}$ exists which solves $\begin{cases} x \equiv 1 \pmod{3}, \\ x \equiv 2 \pmod{9}, \\ x \equiv 3 \pmod{11}. \end{cases}$

7. (a) Show that $8^m \equiv 8 \pmod{28}$ for all $m \in \mathbb{Z}_+$.
   (b) Find all integers $n \geq 2$ that satisfy $8^m \equiv 8 \pmod{n}$ for all $m \in \mathbb{Z}_+$.

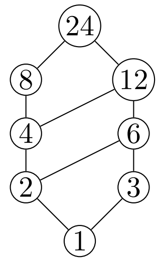
8. How many different "words" (letter sequences) can be formed using all letters in COUSCOUS if identical letters are not allowed to be adjacent?

(SVENSK VERSION PÅ OMSTÅENDE SIDA)

Solutions

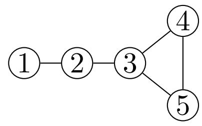
1. Among the $5^3 = 125$ functions, $5 \cdot 4 \cdot 3 = 60$ are injective. **Answer:** 65

2. The set which is ordered is $\{1, 2, 3, 4, 6, 8, 12, 24\}$. **Answer:**



3. A connected graph with one 3-cycle and no other cycles works, since any spanning tree is obtained by deleting one edge of the cycle.

Answer:



4. (a) One hamiltonian cycle is $1, 2, 3, 4, 6, 7, 5, 1$ **Answer:** G is hamiltonian.
   (b) There are odd cycles in G; one is $1, 2, 3, 7, 5, 1$. **Answer:** G is not bipartite.
   (c) Consider the complete bipartite graph isomorphic to $K_{3,3}$ which has vertex bipartition $X = \{1, 3, 6\}$, $Y = \{2, 4, 7\}$. If one subdivides the edge $\{1, 7\}$, and calls the new vertex 5, one obtains the graph G. By (the "easy direction" of) Kuratowski's theorem, G is not planar. **Answer:** G is not planar.

5. The characteristic polynomial is $x^2 - 2x - 3 = (x + 1)(x - 3)$. Hence, the homogeneous part of the solution is $a_n^{\text{hom}} = C_1 \cdot (-1)^n + C_2 \cdot 3^n$ for constants C_1 and C_2.

   In order to find a particular solution, we make the ansatz $a_n^{\text{part}} = An \cdot (-1)^n$. (The naïve ansatz $a_n^{\text{part}} = A \cdot (-1)^n$ would not work since it is a homogeneous solution.) Inserting the ansatz in the recurrence yields

$$A(n + 2) \cdot (-1)^{n+2} = 2A(n + 1) \cdot (-1)^{n+1} + 3An \cdot (-1)^n + 8 \cdot (-1)^n,$$

   which has the unique solution $A = 2$. Thus, the general solution to the recurrence equation is $a_n = (C_1 + 2n) \cdot (-1)^n + C_2 \cdot 3^n$ for constants C_1 and C_2. The initial values translate to the system of equations

$$\begin{cases} 4 &= C_1 + C_2, \\ -2 &= -C_1 + 3C_2 - 2, \end{cases}$$

   so that $C_1 = 3$, $C_2 = 1$. **Answer:** $a_n = (3 + 2n) \cdot (-1)^n + 3^n$

6. (a) Since $\gcd(3, 8) = \gcd(3, 11) = \gcd(8, 11) = 1$, the Chinese remainder theorem informs us that the system has a unique solution modulo $3 \cdot 8 \cdot 11 = 264$. Since $1 \cdot 88 \equiv 1 \pmod 3$, $1 \cdot 33 \equiv 1 \pmod 8$, and $(-5) \cdot 24 \equiv 1 \pmod{11}$ (figure this out using Euclid if necessary), said theorem moreover asserts that the solution is

$$x = 1 \cdot 1 \cdot 88 + 2 \cdot 1 \cdot 33 + 3 \cdot (-5) \cdot 24 = 88 + 66 - 360 \equiv 154 - 96 = 58 \pmod{264}.$$

   Answer: $x = 58 + 264k$, $k \in \mathbb{Z}$

(b) The second congruence means that $x = 2+9k$, $k \in \mathbb{Z}$, which implies $x \equiv 2 \pmod 3$. This contradicts the first congruence. $\square$

7. (a) We induct on m and compute modulo 28. In the base case $m = 1$, the assertion is $8 \equiv 8$, which is true. Now fix $m \geq 2$ and assume by induction that $8^{m-1} \equiv 8$. Then,

$$8^m = 8 \cdot 8^{m-1} \equiv 8 \cdot 8 = 64 = 56 + 8 \equiv 8,$$

as desired. $\square$

(b) If we pretend that the calculations in (a) are carried out modulo n, we notice that they are still true if $56 \equiv 0$, i.e. if n divides 56. On the other hand, if n does not divide 56, $8^2 \not\equiv 8$. Thus, n satisfies the desired condition if and only if n divides 56.

Answer: $n \in \{2, 4, 7, 8, 14, 28, 56\}$

8. Ignoring the restriction on adjacent letters, there are $\binom{8}{2,2,2,2}$ different words. Let A_{C} denote the set of words that contain two adjacent C, and define A_{O}, A_{U}, and A_{S} similarly. We are going to apply PIE, hence need to compute the cardinalities of all possible intersections of these four sets. By symmetry, these cardinalities only depend on how many sets we intersect. Considering adjacent letters as single symbols, such as 'CC', 'OO', etc., we compute

$$\begin{aligned}
|A_{\text{C}}| &= \binom{7}{2,2,2,1}, \\
|A_{\text{C}} \cap A_{\text{O}}| &= \binom{6}{2,2,1,1}, \\
|A_{\text{C}} \cap A_{\text{O}} \cap A_{\text{U}}| &= \binom{5}{2,1,1,1}, \\
|A_{\text{C}} \cap A_{\text{O}} \cap A_{\text{U}} \cap A_{\text{S}}| &= \binom{4}{1,1,1,1} = 4!.
\end{aligned}$$

By PIE, the number we seek is

$$\begin{aligned}
&\binom{8}{2,2,2,2} - 4 \cdot \binom{7}{2,2,2,1} + 6 \cdot \binom{6}{2,2,1,1} - 4 \cdot \binom{5}{2,1,1,1} + 4! \\
&= \frac{7!}{2} - \frac{7!}{2} + 9 \cdot 5! - 2 \cdot 5! + 4! \\
&= 4! \cdot (7 \cdot 5 + 1) \\
&= 24 \cdot 36 \\
&= 864.
\end{aligned}$$

Answer: 864