Linköpings universitet
Matematiska institutionen
Algebra, geometri och diskret matematik

Kurskod: TATA82
Modul: TEN1

Tentamen i TATA82 Diskret matematik

2025-08-21 kl 14.00–19.00

Inga hjälpmedel. Ej räknedosa.

På del A (uppgift 1–3) ska endast svar ges. De ska lämnas på ett gemensamt papper. *Varje uppgift på del A ger högst 1 poäng. Uppgifterna på del B (uppgift 4–8) ger högst 3 poäng per uppgift. Till dessa krävs fullständiga lösningar.*

Godkänt på alla tre kontrollskrivningar *KTR1–3 år 2024 eller 2025 adderar 1 bonuspoäng till totalpoängen. Markera detta genom att skriva ”G” i rutan för uppgift 9 på skrivningsomslaget.*

För betyg 3/4/5 krävs 9/12/15 poäng totalt.

Lösningsförslag finns efter skrivtidens slut på kursens hemsida.

DEL A

1. Låt $a_0 = 1$, $a_1 = 6$ och $a_{n+2} = 5a_{n+1} - 6a_n$ om $n \in \mathbb{N}$. Bestäm a_{100}.
2. Låt $\mathcal{R}$ vara en relation på mängden A. Vad är definitionen av att $\mathcal{R}$ är *reflexiv*?
3. En graf G har följande sekvens av gradtal på hörnen: $0, 1, 1, 2, 3, 3, 4, 6$. Hur många kanter har G?

DEL B

4. (a) Låt d beteckna den största gemensamma delaren till 126 och 308. Beräkna d och två heltal α och β sådana att $126\alpha + 308\beta = d$.

   (b) Finn alla heltalslösningar till den diofantiska ekvationen $126x + 308y = -42$.
5. Låt $n \geq 13$ vara ett heltal. Ur en rad med n olika objekt ska fyra väljas ut så att varje utvalt objekt har minst tre objekt mellan sig och varje annat utvalt objekt. På hur många sätt kan det göras?
6. (a) Antag att n är ett positivt heltal med siffrorna $a_0, a_1, \ldots, a_k$, där a_0 är entalssiffran, a_1 tiotalssiffran, etc. Visa att n är kongruent modulo 11 med sin alternerande siffersumma $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^k a_k$.
   Exempelvis gäller $819251 \equiv 1 - 5 + 2 - 9 + 1 - 8 \pmod{11}$.

   (b) Reducera 36846^{33} modulo 11.
7. (a) Låt $V = \{1, 2, 3, 4, 5\}$, $E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$ och betrakta grafen $G = (V, E)$. Beräkna det kromatiska polynomet $P(G, x)$.

   (b) En graf H har det kromatiska polynomet $P(H, x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x$. Visa att H inte är bipartit.
8. Visa att $\displaystyle\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$ för alla $n \in \mathbb{Z}_+$.

(ENGLISH VERSION ON OPPOSITE PAGE)

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Algebra, geometri och diskret matematik

Course code: TATA82
Module: TEN1

Examination in TATA82 Discrete mathematics

2025-08-21 at 14.00–19.00

No aid. No calculator.

In part A (problems 1–3), only answers shall be given. They are to be handed in on a single sheet of paper. *Each problem in part A is worth 1 point. The problems in part B (problems 4–8) are worth 3 points each. For them, complete solutions are required.*

Having passed all three digital tests *KTR1–3 in 2024 or 2025 adds 1 bonus point to the total score. Indicate this by writing "G" in the box representing problem 9 on the exam cover.*

For grade 3/4/5 is required a total of 9/12/15 points.

After the exam, solutions are available from the course webpage.

PART A

1. Let $a_0 = 1$, $a_1 = 6$ and $a_{n+2} = 5a_{n+1} - 6a_n$ if $n \in \mathbb{N}$. Compute a_{100}.
2. Let $\mathcal{R}$ be a relation on the set A. What is the definition of $\mathcal{R}$ being *reflexive*?
3. A graph G has the following vertex degree sequence: $0, 1, 1, 2, 3, 3, 4, 6$. How many edges does G have?

PART B

4. (a) Let d denote the greatest common divisor of 126 and 308. Compute d and two integers α and β such that $126\alpha + 308\beta = d$.

   (b) Find all integer solutions to the diophantine equation $126x + 308y = -42$.
5. Let $n \geq 13$ be an integer. From a row of n different objects, four are to be chosen in such a way that each chosen object has at least three objects between itself and every other chosen object. In how many ways can it be done?
6. (a) Suppose n is a positive integer with digits $a_0, a_1, \ldots, a_k$, where a_0 is the ones digit, a_1 the tens digit, etc. Show that n is congruent modulo 11 to its alternating digit sum $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^k a_k$.
   For example, it holds that $819251 \equiv 1 - 5 + 2 - 9 + 1 - 8 \pmod{11}$.

   (b) Reduce 36846^{33} modulo 11.
7. (a) Let $V = \{1, 2, 3, 4, 5\}$, $E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\}$, and consider the graph $G = (V, E)$. Find the chromatic polynomial $P(G, x)$.

   (b) A graph H has the chromatic polynomial $P(H, x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x$. Show that H is not bipartite.
8. Show that $\displaystyle\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$ for all $n \in \mathbb{Z}_+$.

(SVENSK VERSION PÅ OMSTÅENDE SIDA)

Solutions

1. The characteristic polynomial of the recurrence is $x^2 - 5x + 6 = (x-2)(x-3)$. Hence, the general solution is $a_n = C_1 \cdot 2^n + C_2 \cdot 3^n$ for constants C_1 and C_2. The initial conditions imply $C_1 = -3$, $C_2 = 4$. **Answer:** $a_{100} = -3 \cdot 2^{100} + 4 \cdot 3^{100}$.

2. That $a\mathcal{R}a$ for all $a \in A$.

3. By the handshake lemma, the number of edges is $\frac{1}{2}(0+1+1+2+3+3+4+6) = 10$.
Answer: 10

4. (a) Applying Euclid's algorithm yields

$$\begin{aligned} 308 &= 2 \cdot 126 + 56, \\ 126 &= 2 \cdot 56 + 14, \\ 56 &= 4 \cdot 14 + 0. \end{aligned}$$

Hence, $\gcd(308, 126) = 14$. Further exploiting the calculations, we find

$$14 = 126 - 2 \cdot 56 = 126 - 2 \cdot (308 - 2 \cdot 126) = 5 \cdot 126 - 2 \cdot 308.$$

Answer: $d = 14$, $\alpha = 5$, $\beta = -2$.

(b) Since $-42 = -3 \cdot 14$, (a) shows that one solution is $x = -15$, $y = 6$. Thus, all solutions are given by $x = -15 + \frac{308}{14}k = -15 + 22k$, $y = 6 - \frac{126}{14}k = 6 - 9k$, $k \in \mathbb{Z}$. Using $m = k - 1$ is perhaps more aesthetic.
Answer: $x = 7 + 22m$, $y = -3 - 9m$, $m \in \mathbb{Z}$.

5. A selection of the described form cuts the remaining $n - 4$ objects into five consecutive segments (although the first and/or the last segment may be empty). The specified condition means that each of the middle three segments contains at least three objects. Thus, letting x_i denote the number of objects in the i:th segment, such a selection can be thought of as a solution to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = n - 4$$

subject to the conditions $x_i \in \mathbb{N}$ for all i, and $x_i \geq 3$ for $i \in \{2, 3, 4\}$. Introducing the new variables $y_i = x_i - 3$ for $i \in \{2, 3, 4\}$, this is equivalent to the equation

$$x_1 + y_2 + y_3 + y_4 + x_5 = n - 13,$$

with each variable being a nonnegative integer. A solution to this equation can be represented by a sequence of $n - 9$ symbols of which $n - 13$ are stars and 4 are bars.
Answer: $\binom{n-9}{4}$

6. (a) The digits being $a_0, a_1, \ldots, a_k$ means

$$\begin{aligned} n &= a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k \\ &\equiv a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + \cdots + a_k \cdot (-1)^k \pmod{11}. \end{aligned}$$

This is the alternating digit sum of n. $\square$

(b) By part (a) and Fermat,

$$36846^{33} \equiv ((6 - 4 + 8 - 6 + 3)^{11})^3 = (7^{11})^3 \equiv 7^3 = 343 \equiv 3 - 4 + 3 = 2 \pmod{11}.$$

Answer: 2

7. (a) Let, for example, $e = \{2, 3\}$. Then, $G - e$ is a 5-vertex path and $G \cdot e$ is a 3-cycle with one of its vertices connected to a fourth vertex (the vertex 5). In both cases we can compute the number of proper k-colourings (for $k \in \mathbb{N}$) by considering the vertices one at a time; in $G - e$, consider them in the order $2, 1, 4, 3, 5$ (from one endpoint of the path to the other), and in $G \cdot e$, use any order in which 5 comes last. This gives the conclusion that $G - e$ has $k(k-1)(k-1)(k-1)(k-1)$ proper k-colourings, whereas $G \cdot e$ has $k(k-1)(k-2)(k-1)$. By deletion-contraction,

$$\begin{aligned} P(G, x) &= P(G - e, x) - P(G \cdot e, x) \\ &= x(x-1)^4 - x(x-1)^2(x-2) \\ &= x(x-1)^2(x^2 - 2x + 1 - (x-2)) \\ &= x(x-1)^2(x^2 - 3x + 3). \end{aligned}$$

Answer: $P(G, x) = x(x-1)^2(x^2 - 3x + 3)$.

(b) A graph is bipartite if and only if it has a proper 2-colouring. The number of proper 2-colourings of H is $P(H, 2) = 32 - 5 \cdot 16 + 10 \cdot 8 - 10 \cdot 4 + 4 \cdot 2 = 0$. Hence, H cannot be properly 2-coloured. □

8. The right hand side of the given identity is the number of n-element subsets of $\{1, 2, \ldots, 2n\}$. For a fixed $j \in \{0, 1, \ldots, n\}$, the number of such subsets that contain exactly j elements from $\{1, 2, \ldots, n\}$ and, hence, $n - j$ elements from $\{n+1, n+2, \ldots, 2n\}$ is $\binom{n}{j}\binom{n}{n-j}$. Thus, the number of n-element subsets of $\{1, 2, \ldots, 2n\}$ is also equal to

$$\sum_{j=0}^{n} \binom{n}{j}\binom{n}{n-j},$$

which, by the symmetry property $\binom{n}{n-j} = \binom{n}{j}$, is equal to the left hand side of the identity. □