EXERCISES IN DISCRETE MATHEMATICS

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Problems

P.1 Set notation

P.1.1 Define $A = \{\{1, 2\}, 2, \heartsuit\}$ and $B = \{1, 2\}$. Compute
(a) $A \cup B$ (b) $A \cup \{B\}$ (c) $A \times B$ (d) $A \times \{B\}$ (e) $A \setminus B$ (f) $A \setminus \{B\}$.

P.1.2 Define A and B as in P.1.1. Determine whether the following assertions are true or false:
(a) $1 \in A$ (b) $1 \in B$ (c) $\{1, 2\} \in A$ (d) $\{1, 2\} \in B$ (e) $\emptyset \in B$.

P.1.3 Define A and B as in P.1.1. Determine whether the following assertions are true or false:
(a) $B \subseteq A$ (b) $A \subseteq B$ (c) $\{1, 2\} \subseteq A$ (d) $\{1, 2\} \subseteq B$ (e) $\emptyset \subseteq A$.

P.1.4 Let A and B be as in P.1.1. Compute the power sets $\mathcal{P}(A)$, $\mathcal{P}(B)$, $\mathcal{P}(A \cap B)$, and $\mathcal{P}(A \cap \{B\})$.

P.1.5 Write down a list of all elements of the given set:

(a) $\{m \in \mathbb{Z} : |m| < 3\}$,

(b) $\{(x, y) \in \mathbb{N}^2 : y = 7 - 3x\}$,

(c) $\{(p, q) \in \mathbb{Z}_+ \times \mathbb{Z} : pq = -8\}$.

P.2 Induction and recurrence

P.2.1 Show that $1 + 3 + \cdots + (2n + 1) = (n + 1)^2$ for all $n \in \mathbb{N}$ by induction on n.

P.2.2 Show that $\displaystyle\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$ by induction on n.

P.2.3 Use induction on n to prove the *Bernoulli inequality* which states that $(1 + x)^n \geq 1 + nx$ for all real $x \geq -1$ and all $n \in \mathbb{Z}_+$.

P.2.4 Prove that $3^n > 2n^2 + 8$ for all integers $n \geq 3$ by induction on n.

P.2.5 "3's and 7's" is a listenable song by Queens of the Stone Age. Prove that every integer strictly larger than 11 can be written as a sum of 3's and 7's.
[hint] [walkthrough]

P.2.6 One cuts the plane into regions by drawing a finite number of straight lines. Two regions are *adjacent* if they are separated by exactly one line. Prove, using induction on the number of lines, that it is possible to assign to each region either the colour black or the colour white so that no two adjacent regions receive the same colour.
[hint] [spoiler]

P.2.7 One cuts the plane into regions by drawing a finite number of straight lines. Use induction on n to prove that the number of regions is at most $\frac{n(n+1)}{2}+1$, where n is the number of lines.
[hint] [spoiler] [walkthrough]

P.2.8 You have 3^n marbles that look identical. One of them, however, is fake and slightly lighter than the others. Prove that you can find the fake marble using at most n weighings on a balance.
[hint]

P.2.9 Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain two successive integers. Prove that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. What are the starting values a_1 and a_2?
[hint] [walkthrough]

P.2.10 Express the numbers a_n defined in P.2.9 in terms of the Fibonacci numbers F_n (denoted f_n in Rosen's book).

P.2.11 Let $a_0 = 0$ and $a_n = \sqrt{2a_{n-1}} + a_{n-1} + \frac{1}{2}$ if $n \in \mathbb{Z}_+$. Compute a_n for a few values of n. Then, guess a formula and prove your guess using induction.

P.2.12 Define a_n for $n \in \mathbb{N}$ by the recurrence $a_n = 6a_{n-1} - 9a_{n-2} + 4n - 12$ if $n \geq 2$ and the initial values $a_0 = 2$, $a_1 = 4$. Use induction to show that $a_n = (2-n)3^n + n$ for all $n \in \mathbb{N}$.

P.2.13* Define d_n for $n \in \mathbb{N}$ by the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$ if $n \geq 2$ and the initial values $d_0 = 1$, $d_1 = 0$. Use induction to show that $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ for all $n \in \mathbb{N}$.

P.2.14 Solve the recurrence $a_{n+2} = 4(a_{n+1} - a_n)$, $n \in \mathbb{N}$.

P.2.15 Solve the initial value problem $a_0 = 4$, $a_1 = 1$, $a_2 = 2$, $a_{n+3} = a_{n+2} + a_{n+1} - a_n$, $n \in \mathbb{N}$.

P.2.16 Let a_n denote the number of ways to write n as a sum where every term is 2 or 4, and the order of the terms matters. (For example, $a_6 = 3$ since $6 = 2+2+2 = 2+4 = 4+2$.) Find a formula for a_n, $n \in \mathbb{N}$, in terms of the Fibonacci numbers F_n.
[hint]

P.2.17 How many "words" of length $n \in \mathbb{N}$ can be constructed using only the letters A, B, and C, if an A is never allowed to be adjacent to another A?
[hint] [spoiler]

P.2.18 Let $a_n = \sum_{j=0}^{n} j^2$ for $n \in \mathbb{N}$. Find a recurrence relation for a_n and solve it using the methods you have learnt for solving such relations, thereby reproving the formula from P.2.2.
[walkthrough]

P.2.19 Solve the recurrence $a_{n+1} = 3a_n + 3^n + n$, $n \in \mathbb{N}$.

P.2.20 Solve the initial value problem $a_0 = a_1 = 0$, $a_{n+2} = a_{n+1} + 2a_n + 2n$, $n \in \mathbb{N}$.

P.3 Combinatorics

P.3.1 A traditional Swedish car license plate has three letters followed by three digits, such as ABC123. The possible combinations were running out, and in 2019 we started using a new standard in which the last symbol is a letter instead of a digit, such as BAB22D. In both standards, there are 23 letters allowed (I, Q, V, Å, Ä, and Ö are not), but the last letter in the new standard is not allowed to be O (which is too similar to the digit 0). How many additional license plates became possible by introducing the new standard? How many were possible before?[1]

P.3.2 How many traditional license plates (see P.3.1) contain no symbol more than once?

P.3.3 In a tournament, m teams participate. Every team plays every other team once. How many games are played?

P.3.4 Eight people form a single queue at the store. In how many ways can they do it?

P.3.5 Eight people split into two non-empty queues to different counters at the store. In how many ways can they do it?
[spoiler]

P.3.6 Eight people split into two queues of equal size to different counters at the store. In how many ways can they do it?

P.3.7 At a dinner, m women and m men are to be seated around a circular table in such a way that men and women alternate. In how many ways can they be seated? (Two arrangements that differ only by a rotation of the table are considered to be equal.)

P.3.8 How many four-letter "words" can be formed by using letters from HIGHLIGHT?
[hint]

[1] In reality, not all combinations are allowed. One example is LSD111. There are worse examples. Ignore these restrictions.

P.3.9 In a two-person game of five-card poker (52-card deck, no wild cards), you are dealt four of a kind: four jacks and the ace of spades. What is the probability that your opponent was dealt a better hand?[2]
[hint]

P.3.10 How many surjective functions $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 7\}$ exist?
[walkthrough]

P.3.11* How many surjective functions $f : \{1, 2, \ldots, n + 2\} \to \{1, 2, \ldots, n\}$ exist?
[hint]

P.3.12 Use the binomial theorem to prove that $5^n = \sum_{k=0}^{n} \binom{n}{k} 4^k$ for all $n \in \mathbb{N}$.

P.3.13 Prove that $5^n = \sum_{k=0}^{n} \binom{n}{k} 4^k$ for all $n \in \mathbb{N}$ by counting, in two different ways, the number of ways to colour n objects using at most five colours.

P.3.14 Use the multinomial theorem to prove that $3^n = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{n}{k, m, n-m-k}$ holds for all $n \in \mathbb{N}$. Can you also give a combinatorial proof, by counting something in two different ways?
[hint]

P.3.15 How many "words" can be formed by using all letters from HIGHLIGHT?

P.3.16 How many "words" can be formed by using all letters from HIGHLIGHT if it is forbidden to have two adjacent "H"?

P.3.17 Fifteen different (but equally entertaining) toys are distributed fairly among five children. In how many ways can it be done?

P.3.18 How many solutions to $x_1 + x_2 + x_3 + x_4 = 14$ satisfy $x_i \in \mathbb{N}$ for all i?

P.3.19 How many solutions to $x_1 + x_2 + x_3 + x_4 \leq 14$ satisfy $x_i \in \mathbb{N}$ for all i?
[spoiler]

P.3.20 In how many ways can 20 identical objects be distributed into four different boxes in such a way that no box remains empty and at least three objects are placed in box number two?

P.3.21 How many 5-element subsets of $\{1, 2, \ldots, n\}$, $n \geq 5$, contain no pair of consecutive elements?
[hint] [walkthrough]

[2] The only hands that beat you are a better four of a kind, or a straight flush. If you wonder what these words mean, ask your teacher, your friend, or your favourite search engine.

P.3.22 How many increasing[3] surjective functions $f : \{1, 2, \ldots, n\} \to \{1, 2, 3, 4, 5\}$ exist?

P.3.23* Prove that there are exactly $(k+1)^n$ different k-tuples $(S_1, S_2, \ldots, S_k)$ of sets that satisfy $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq \{1, 2, \ldots, n\}$.
[hint] [walkthrough]

P.3.24 In a tournament, at least two teams participate. Every team plays every other team once. Prove that at any given time, there are two teams that have played the same number of games.

P.3.25 Prove that if 451 different three-digit numbers are selected, there are two among them that sum to 1099.
[spoiler]

P.3.26 Show that among seven real numbers we can always find two, x_1 and x_2, such that $0 \leq \arctan x_1 - \arctan x_2 < \dfrac{\pi}{6}$.
[spoiler] [walkthrough]

P.3.27 In how many permutations of $\{1, 2, \ldots, n\}$ is 1 not adjacent to 2?

P.3.28 Use inclusion-exclusion to compute the number of permutations of $\{1, 2, \ldots, n\}$ in which 1 is neither adjacent to 2 nor to 3.[4]

P.3.29 There were 80 students enrolled in the Indiscreet mathematics course. The examination was divided into three exams: TEN1, TEN2, and TEN3. In order to pass the course, one needed to pass all three exams. A total of 59 students passed TEN1, 48 passed TEN2, and 60 passed TEN3. Moreover, 40 passed both TEN1 and TEN2, 50 passed both TEN1 and TEN3, and 38 passed both TEN2 and TEN3. Six students failed every exam. How many passed the course?

P.3.30 How many permutations of the 26-letter alphabet $\{\text{A}, \text{B}, \ldots, \text{Z}\}$ do not contain any of the substrings BOWL, GARBO, or OWLET?
[hint]

P.3.31 How many positive integers $N \leq 6300$ satisfy that at least one of $\dfrac{N}{7}$, $\dfrac{N}{9}$, and $\dfrac{N}{10}$ is an integer?

P.3.32 How many solutions to $x_1 + x_2 + x_3 + x_4 \leq 14$ satisfy $x_i \in \mathbb{N}$ and $x_i \leq 5$ for all i?
[walkthrough]

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[3] Recall that f is *increasing if* $a \leq b \Rightarrow f(a) \leq f(b)$.

[4] Once you see the (simplified) result, it looks strikingly simple. Can you find a direct combinatorial proof? (Alternatively, if you *did* see a direct proof immediately, you should now also prove it using inclusion-exclusion.)

P.3.33* A permutation of $\{1, 2, \ldots, n\}$ is *well-mixed* if k is never immediately followed by $k+1$ for any $k \in \{1, 2, \ldots, n-1\}$. Prove that the number of well-mixed permutations of $\{1, 2, \ldots, n\}$ is $(n-1)!\sum_{k=0}^{n-1}(-1)^k \frac{n-k}{k!}$.
[hint]

P.4 Number theory

P.4.1 List all divisors of 45.

P.4.2 List all primes $p < 50$.

P.4.3 Prove that $6|(n^3 - n)$ for every $n \in \mathbb{Z}$.

P.4.4 Prove that if p is a prime and $p+1$ is a square, then $p = 3$.

P.4.5 Use prime factorization to compute $\gcd(693, 990)$ and $\operatorname{lcm}(693, 990)$.

P.4.6 Suppose $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, where the p_i are pairwise distinct primes and the k_i are positive integers. How many divisors does n have? How many are positive?

P.4.7 How many positive integers $N \leq 2000$ satisfy that at least one of $\frac{N}{6}$, $\frac{N}{8}$, and $\frac{N}{15}$ is an integer?
[walkthrough]

P.4.8 Use Euclid's algorithm to compute $\gcd(444, 210)$.

P.4.9 Show that $\gcd(3n^2 + n - 7, n^2 - 2) = 1$ for all $n \in \mathbb{N}$.
[hint] [walkthrough]

P.4.10 Find $\alpha, \beta \in \mathbb{Z}$ such that $444\alpha + 210\beta = \gcd(444, 210)$.

P.4.11 Solve the diophantine equation $210x + 444y = -18$.

P.4.12 Solve the diophantine equation $210x + 444y = -19$.

P.4.13 Find all integer solutions to $51x + 14y = 2$ that satisfy $x \leq 10$ and $y \leq 10$.

P.4.14 Solve the diophantine equation $10x + 7y + 14z = 13$.

P.4.15 Solve the diophantine equation $x^2 - 4y^2 = 13$.
[hint] [spoiler]

P.5 Relations and posets

P.5.1 Consider the relation $\mathcal{R} = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3),(4,3),(4,4)\}$ on the set {1,2,3,4}. Is $\mathcal{R}$ (i) reflexive? (ii) symmetric? (iii) antisymmetric? (iv) transitive?

P.5.2 Consider the relation $\mathcal{R} = \{(1,2),(1,3),(1,4),(2,2),(2,3),(3,3),(4,4)\}$ on the set {1,2,3,4}. Is $\mathcal{R}$ (i) reflexive? (ii) symmetric? (iii) antisymmetric? (iv) transitive?

P.5.3 How many relations on $\{a,b,c\}$ are (i) reflexive? (ii) symmetric? (iii) antisymmetric?

P.5.4 Let $\sim$ mean "has the same colour as". Prove that $\sim$ is an equivalence relation on the set of your (single coloured) socks. What are the equivalence classes?

P.5.5 Define a relation $\sim$ on $\mathbb{C}$ by declaring $x \sim y$ if $|x| = |y|$. Show that $\sim$ is an equivalence relation and describe the equivalence classes.

P.5.6 Let $X = \{1,2,\ldots,9\}$ and denote by $\mathcal{P}(X)$ the set of all subsets of X. Define a relation $\mathcal{R}$ on $\mathcal{P}(X)$ by taking $S\,\mathcal{R}\,T$ to mean that $|S| = |T|$. Show that $\mathcal{R}$ is an equivalence relation. Describe the equivalence classes. How many sets belong to the equivalence class $[\{1,2,3\}]$?

P.5.7 Let $x \bowtie y$ mean that xy is a square. (For example, $8 \bowtie 18$ since $8 \cdot 18 = 144 = 12^2$.) Show that $\bowtie$ is an equivalence relation on $\mathbb{Z}_+$. List a few of the smallest elements in the equivalence class [3].

[walkthrough]

P.5.8 Show that $\bowtie$ (as described in P.5.7) is not an equivalence relation on $\mathbb{N}$.

P.5.9 Let ℓ be a line in $\mathbb{R}^4$ which contains the origin. Define a relation $\frown$ on $\mathbb{R}^4$ by taking $\mathbf{u} \frown \mathbf{v}$ to mean that $\mathbf{u} - \mathbf{v} \in \ell$. Show that $\frown$ is an equivalence relation. Can you describe the equivalence classes?

P.5.10 The following "proof" that a symmetric and transitive relation is necessarily reflexive is incorrect. Find the flaw! *"Proof": Suppose $\mathcal{R}$ is symmetric and transitive on the set X. Pick $x \in X$. We want to show that $x\mathcal{R}x$. Choose $y \in X$ such that $x\mathcal{R}y$. Since $\mathcal{R}$ is symmetric, $y\mathcal{R}x$. Since $x\mathcal{R}y$ and $y\mathcal{R}x$, transitivity shows $x\mathcal{R}x$. Hence, $\mathcal{R}$ is reflexive.*

P.5.11 How many relations on $\{1,2,\ldots,n\}$ are both equivalence relations and partial orders?

P.5.12 Suppose X is a finite set. Let $\mathcal{P}_{\text{even}}(X)$ denote the set of all subsets of X that contain an even number of elements. Prove that the subset relation $\subseteq$ is a partial order on $\mathcal{P}_{\text{even}}(X)$. Describe the maximal and minimal elements. Is there a maximum? a minimum?

P.5.13 Draw the Hasse diagram of the poset described in P.5.12 in the cases $X = \{1,2,3,4\}$ and $X = \{1,2,3,4,5\}$.

P.5.14 Let $n \in \mathbb{N}$. The set of all positive divisors of n is a poset when ordered by the divisibility relation. Let us denote this poset D_n. Draw the Hasse diagram of D_{75}.

P.5.15 Let $n \in \mathbb{N}$. Prove that the set of all divisors of n is not a poset under the divisibility relation.
[walkthrough]

P.5.16 The *lexicographic order* on $\mathbb{N} \times \mathbb{N}$ is defined by letting $(x_1, x_2) \preceq (y_1, y_2)$ if either $x_1 < y_1$ or else $x_1 = y_1$ and $x_2 \leq y_2$. Prove that $\preceq$ is a total order on $\mathbb{N} \times \mathbb{N}$.

P.5.17 Suppose P and Q are two disjoint sets (i.e. $P \cap Q = \emptyset$) that are partially ordered with order relations $\leq_P$ and $\leq_Q$, respectively. Define the *ordinal sum* $P \oplus Q$ as the set $P \cup Q$ equipped with the relation $\leq$ given by $x \leq y$ if either (i) $x, y \in P$ and $x \leq_P y$, or (ii) $x, y \in Q$ and $x \leq_Q y$, or (iii) $x \in P$ and $y \in Q$. Show that $P \oplus Q$ is a poset. How would you construct the Hasse diagram of $P \oplus Q$ starting from the diagrams of P and Q, if P and Q are finite?

P.5.18 Prove that every finite, nonempty lattice has a minimum and a maximum. Also, give an example of an infinite lattice which neither has a minimum nor a maximum.
[spoiler]

P.5.19 Order $\{1, 2, \ldots, n\}$ by divisibility. Prove that if $n \geq 3$, then the resulting poset is not a lattice.
[hint]

P.5.20 Use your solution to P.5.14 to verify that D_{75} is a lattice. More generally, prove that D_n is a lattice for every $n \in \mathbb{N}$.
[walkthrough]

P.5.21 Prove that $P \oplus Q$ (as defined in P.5.17) is a lattice if P and Q are.

P.6 Modular arithmetic

P.6.1 Reduce 2^{43} modulo 15.

P.6.2 Prove that $3^{4n+1} \equiv 3 \pmod{15}$ for all $n \in \mathbb{N}$. Does it follow that $3^{4n} \equiv 1 \pmod{15}$? Why, or why not?

P.6.3 Reduce 3^{43} modulo 15, for example by glancing at P.6.2.

P.6.4 The method of *casting out nines* exploits that an integer n is divisible by 9 if and only if the sum of digits of n (when represented in base 10) is divisible by 9. For example, 569754 is divisible by 9 since $5 + 6 + 9 + 7 + 5 + 4 = 36$ is divisible by 9. Prove that this method is valid.

P.6.5 Find the last two digits of the number 7^{38}.

P.6.6 Prove that there is no $x \in \mathbb{Z}$ such that $x^3 - 3x + 1$ is divisible by 7.
[walkthrough]

P.6.7 One of 14 and 15 is invertible modulo 51. Which one? Compute the inverse.

P.6.8 Find all $x \in \mathbb{Z}$ that satisfy $14x \equiv 7 \pmod{51}$.
[spoiler]

P.6.9 You want to plant your flowers in an array-shaped pattern. If you plant nine flowers in each row, seven flowers are left over. If you plant eight flowers in each row, three flowers are left over. The number of flowers is more than 100, but less than 150. How many do you have?
[spoiler]

P.6.10 Find all integer solutions to $\begin{cases} x \equiv 0 & \pmod{15}, \\ x \equiv 9 & \pmod{16}, \\ x \equiv 7 & \pmod{49}. \end{cases}$
[hint]

P.6.11 Find all integer solutions to $x^4 + 2x - 8 \equiv 0 \pmod{60}$.
[hint] [spoiler] [walkthrough]

P.6.12 Compute the remainder when 13^{3002} is divided by 31.
[hint]

P.6.13 The universe is approximately $5 \cdot 10^{12}$ days old. What day of the week will it be exactly $5 \cdot 10^{12}$ days from now?

P.6.14 Reduce $1^{10} + 2^{100} + 3^{1000} + \cdots + 10^{10^{10}}$ modulo 11.

P.6.15 Find all $x \in \mathbb{Z}$ that satisfy $x^{67} \equiv 3 \pmod{23}$.

P.6.16* Prove the version of Fermat's little theorem which states that $a^p \equiv a \pmod{p}$ for any prime p and $a \in \mathbb{Z}_+$ by expanding $a^p = (1 + 1 + \cdots + 1)^p$ using the multinomial theorem and looking at the individual terms modulo p.
[spoiler]

P.6.17 In an RSA cipher, Bob has the public key $(69, 5)$. Alice wants to encrypt the message "7" and send it to Bob. What is the ciphertext?

P.6.18 In an RSA cipher, Bob's private key is 7 and the public key is $(55, 23)$. Bob receives the encrypted message "8". What is the plaintext?

P.6.19 In an RSA cipher, Bob's public key is $(91, 29)$. Eve intercepts the encrypted message "7" which was sent to Bob. What is the plaintext?
[hint] [walkthrough]

P.7 Graphs

P.7.1 Draw a (simple) graph with the given degree sequence or prove that none exist:
(a) $0, 1, 3, 3, 4, 4$ (b) $0, 1, 3, 3, 3, 4, 4$ (c) $0, 1, 3, 3, 4, 5$.

P.7.2 How many edges does the complete graph K_n contain?

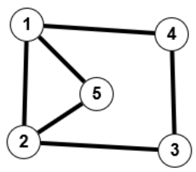
P.7.3 How many edges does a k-regular graph with n vertices contain?

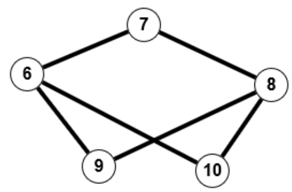
P.7.4 How many vertices does a k-regular graph with ϵ edges contain?
[spoiler]

P.7.5 Prove that an $(n-2)$-regular graph on n vertices exists if and only if n is even.

P.7.6 In a connected simple graph on six vertices, five of the vertices have pairwise different degrees. Which degree does the sixth vertex have?
[walkthrough]

P.7.7 Find an isomorphism between the two graphs or prove that they are not isomorphic.

$G_1 =$ 

$G_2 =$ 

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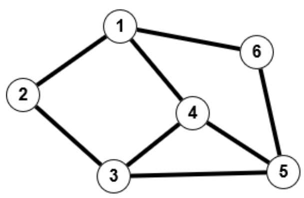
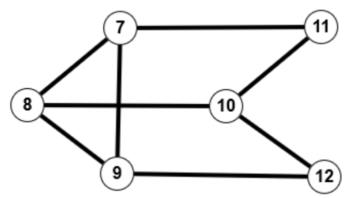
$G_1 =$ 

$G_2 =$ 

P.7.9 How many pairwise non-isomorphic simple graphs on eight vertices and three edges are there?

P.7.10 Which of the graphs in P.7.7 and P.7.8 contain eulerian circuits? eulerian paths? hamiltonian circuits? hamiltonian paths? (Either prove nonexistence or indicate an explicit circuit or path.)

P.7.11 Consider the graph $G = (V, E)$ which has vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edge set $E = \{\{1, 2\}, \{1, 4\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{2, 6\}, \{3, 4\}, \{3, 6\}, \{4, 5\}, \{5, 6\}\}$. Prove that G is hamiltonian, but that removing the edge $\{2, 3\}$ would make it non-hamiltonian.
[spoiler]

P.7.12 Let G be as in P.7.11. Prove that G is not eulerian. How many (and which?) edges would you need to add to G in order to make it eulerian?

P.7.13 The *n-cube* is the graph whose vertices are all length n bit sequences (i.e., all sequences $a_1 a_2 \cdots a_n$ where $a_i \in \{0, 1\}$ for all i), and two such sequences form an edge if and only if they differ in exactly one bit. For which n is the n-cube eulerian? (And, why is it called a "cube"?)

P.7.14 Suppose G is a regular simple graph on an even number, at least 4, of vertices. Prove that either G or the complement graph $\overline{G}$ (or both) is hamiltonian.[5]
[hint] [walkthrough]

P.7.15* Prove that the n-cube (see P.7.13) is hamiltonian for all integers $n \geq 2$.[6]
[hint] [spoiler]

P.7.16 How many 3-cycles are subgraphs of the complete graph K_n? How many 4-cycles?
[spoiler]

P.7.17 For each of the graphs appearing in P.7.7, P.7.8, and P.7.27, determine whether it is bipartite.

P.7.18 How many 4-cycles are subgraphs of the complete bipartite graph $K_{m,n}$? How many 6-cycles?

P.7.19 A collection of n straight lines cuts the plane into regions. Construct a graph whose vertices are the regions, and let two regions form an edge if and only if they are separated by exactly one line. P.2.6 shows that this graph is bipartite. Give a different proof, by arguing directly that the graph does not contain odd cycles.
[hint] [walkthrough]

P.7.20 A forest on three trees has twelve edges. How many vertices are there?

P.7.21 In a certain tree, the non-leaves have the following degrees: $2, 3, 4, 4, 5$. How many leaves are there?
[hint]

P.7.22 Suppose T is an n-vertex tree. How many subgraphs of T are forests on exactly n vertices and 3 trees?

P.7.23 Suppose T is an n-vertex tree. How many simple graphs contain T as a spanning tree?

P.7.24 Prove that a graph G is a tree if and only if the following two conditions hold:

(i) Adding an edge to G always results in a graph which contains a cycle that involves the new edge.

(ii) Removing an edge from G always results in a disconnected graph.

[spoiler]

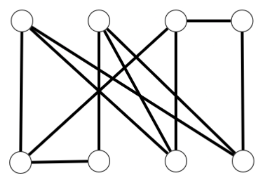
[5] The result is still true if G has an odd number of vertices, but that is harder to prove.

[6] Hamiltonian cycles in the n-cube are called *Gray codes* and have real-life applications. Some are indicated in Rosen's book. Use your favourite search engine to find more.

P.7.25 How many regions can exist in a planar embedding of a simple planar graph with vertex degree sequence $3, 3, 4, 4, 5, 5$?

P.7.26 How many regions can exist in a planar embedding of a 2-regular simple graph on seven vertices?

P.7.27 Show that the following graph is not planar:



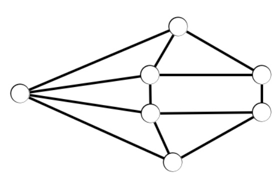
[hint]

P.7.28 Suppose G is a simple graph in which every vertex degree is at least 6. Prove that G is not planar.
[spoiler] [walkthrough]

P.7.29* Suppose G is a simple, planar graph on at least 11 vertices. Prove that the complement graph $\overline{G}$ is not planar.
[hint] [spoiler]

P.7.30* Suppose n straight lines in the plane have the property that no two are parallel and no three intersect in a point. Use Euler's formula[7] for planar graphs to prove that the number of regions separated by the lines is $\frac{n(n+1)}{2} + 1$. (Tip: It is convenient to "bend" the parts of the lines that are "far away" so that they meet in a single point. Then consider the line segments and intersection points as edges and vertices of a plane embedding of a graph.)
[spoiler]

P.7.31 Show that the following graph has chromatic number 4:



[spoiler]

P.7.32 Prove that if the maximum vertex degree of a graph G is k, then G has a proper $(k + 1)$-colouring.
[walkthrough]

[7] You could also use induction directly, as in P.2.7, but that is illegal here.

P.7.33 Prove that if G is a graph with chromatic number k, then G has at least $\binom{k}{2}$ edges.
[spoiler]

P.7.34 Compute the chromatic numbers and polynomials of the graphs in P.7.7.

P.7.35 Compute the chromatic polynomial of a forest with n vertices and m trees.
[hint] [spoiler]

P.7.36 Use induction on n to prove that an n-cycle C_n, $n \geq 3$, has chromatic polynomial $P(C_n, x) = (x-1)^n + (-1)^n(x-1)$.
[hint]

P.7.37 Let G be a graph. Define the *cone* $\mathcal{C}(G)$ to be the graph obtained from G by adding one extra vertex $v \notin V(G)$ and extra edges that connect v to every vertex of G. Prove that $P(\mathcal{C}(G), x) = xP(G, x-1)$.

P.7.38* Let G be a graph. Define the *suspension* $\mathcal{S}(G)$ to be the graph obtained from G by adding two extra vertices $v_1, v_2 \notin V(G)$ and extra edges that connect v_1 to every vertex of G and v_2 to every vertex of G. (Thus, v_1 and v_2 are *not* connected in $\mathcal{S}(G)$.) Prove that $P(\mathcal{S}(G), x) = xP(G, x-1) + x(x-1)P(G, x-2)$.
[walkthrough]

P.7.39 Use the results of P.7.36 and P.7.37 to compute the chromatic polynomial of the *wheel graph* W_n which consists of n vertices that form an n-cycle and one extra vertex (the *hub*) which is connected to all other vertices (the edges containing the hub are the *spokes*).

Hints

H.2.5 Verify the statement for 12, 13, 14. Then use strong induction.

H.2.6 Assume you have a colour assignment that works for a bunch of lines. Can you modify it to work after another line is drawn?

H.2.7 How many new regions can appear when you draw a new line?

H.2.8 Divide the marbles into three piles of equal size and compare two of the piles on the balance. What can you conclude?

H.2.9 Such a subset either contains n or it does not. How many do, and how many do not?

H.2.16 Considerations become simpler if you think of sums of 1's and 2's that sum to $n/2$ for even n.

H.2.17 Find a recurrence and solve it.

H.3.8 It is probably easiest to split into four cases, depending on how letters repeat. Representatives of each case would be HLHH, GIGI, HIGH, and HILT.

H.3.9 Remember that your opponent does not have any of *your* cards.

H.3.11 Either one element in the range is the value of three different inputs, or two elements are the values of two each.

H.3.14 $3 = 1 + 1 + 1$

H.3.21 Reformulate as a "stars and bars" problem.

H.3.23 What could $(k+1)^n$ count? Can you relate that to the k-tuples in the problem?

H.3.30 A permutation that contains both GARBO and OWLET automatically contains BOWL.

H.3.33 If you let A_i denote the set of permutations that have i immediately followed by $i+1$, you want to consider $n! - |A_1 \cup A_2 \cup \cdots \cup A_{n-1}|$.

H.4.9 Euclid

H.4.15 You *could* solve the diophantine linear equation $a - 4b = 13$ and look for solutions that are squares. Then you would probably find the correct answer, but to argue that there are no other solutions could be technical. A more straightforward approach exists.

H.5.19 You could use P.5.18.

H.6.10 CRT

H.6.11 CRT

H.6.12 31 is a prime.

H.6.19 Find Bob's private key.

H.7.14 Use Dirac's (or Ore's) theorem.

H.7.15 Induct on n.

H.7.19 Walking along an edge in the graph corresponds to crossing a line in the plane.

H.7.21 Use the handshake lemma.

H.7.27 Use the "easy direction" of Kuratowski's theorem.

H.7.29 You may assume that G has *exactly* 11 vertices. (Why is it enough to consider that case?) Bound the number of edges.

H.7.35 Colour each tree separately. Colour the vertices of a tree in order so that each vertex is adjacent to at most one already coloured vertex.

H.7.36 Use deletion-contraction.

Spoilers

S.2.6 What happens if you swap all colours on one side of the new line?

S.2.7 A line intersects other lines in at most $n-1$ points. Thus, when line number n is drawn, at most n new regions appear.

S.2.17 Prove that $a_{n+2} = 2a_{n+1} + 2a_n$ if a_n is the number that you seek. For the initial conditions, do not forget that the empty word is one word.

S.3.5 They could do it by first forming one queue and then cutting that queue at one of seven possible positions.

S.3.19 Introduce the new variable $x_5 = 14 - (x_1 + x_2 + x_3 + x_4)$.

S.3.25 $1099 = 100 + 999 = 101 + 998 = \cdots$

S.3.26 Cut the range of arctan into pigeonholes.

S.4.15 Factorize the left hand side. If the right hand side is a product of two integers, what could they be?

S.5.18 Suppose there are two different maximal elements. Use the lattice property to obtain a contradiction.

S.6.8 You could use P.6.7.

S.6.9 Use the Chinese remainder theorem to compute the number of flowers modulo 72.

S.6.11 Find the solutions modulo 3, 4 and 5. Then use CRT.

S.6.16 Argue that if p is a prime, then $\binom{p}{k_1, \ldots, k_a}$ is almost always divisible by p, the exception being if some k_i is equal to p. What is the value then, and how many such terms are there?

S.7.4 Shake hands, or use P.7.3.

S.7.11 Without $\{2, 3\}$ you could remove two vertices to separate the graph into three components.

S.7.15 For the induction step, think of the n-cube as having two layers, both being isomorphic to the $(n-1)$-cube: the bottom layer where the last bit of a vertex is always zero, and the top layer where the last bit is always one. If the bottom layer has a hamiltonian circuit, follow it, but skip its last edge. Then, move up and follow the same circuit backwards. Finally, go down again.

S.7.16 The number of m-cycles is the number of m-subsets of $\{1, 2, \ldots, n\}$ times the number of ways to arrange m elements in a cycle.

S.7.24 Prove that (i) is equivalent to G being connected and (ii) to G being acyclic or disconnected (or both).

S.7.28 Assume $G = (V, E)$ is planar and derive a contradiction. Use the handshake lemma to show $|V| \leq |E|/3$. Count edge-region pairs to show $3f \leq 2|E|$, where f is the number of regions of a planar embedding of G. Then apply Euler's formula.

S.7.29 By counting edge-region pairs, deduce $3f \leq 2|E|$, where f is the number of regions of a planar embedding of $G = (V, E)$. With $|V| = 11$, what does Euler then tell you about $|E|$? How may edges does $\overline{G}$ have?

S.7.30 Let $G = (V, E)$ be the graph constructed by following the tip. Argue that $|E| = n^2$ and $|V| = 1 + \binom{n}{2}$.

S.7.31 There is no 3-colouring (why?). Find a 4-colouring.

S.7.33 Consider a vertex colouring with as few colours at possible. Then $\binom{k}{2}$ is the number of colour pairs.

S.7.35 If a tree has k vertices, use the hint to show that the chromatic polynomial of the tree is $x(x-1)^{k-1}$.

Answers

A.1.1

(a) $\{\{1,2\},1,2,\heartsuit\}$

(b) $\{\{1,2\},2,\heartsuit\}$

(c) $\{(\{1,2\},1),(\{1,2\},2),(2,1),(2,2),(\heartsuit,1),(\heartsuit,2)\}$

(d) $\{(\{1,2\},\{1,2\}),(2,\{1,2\}),(\heartsuit,\{1,2\})\}$

(e) $\{\{1,2\},\heartsuit\}$

(f) $\{2,\heartsuit\}$

A.1.2 (a) false (b) true (c) true (d) false (e) false

A.1.3 (a) false (b) false (c) false (d) true (e) true

A.1.4

$$\begin{aligned}
\mathcal{P}(A) &= \{\emptyset,\{\{1,2\}\},\{2\},\{\heartsuit\},\{\{1,2\},2\},\{\{1,2\},\heartsuit\},\{2,\heartsuit\},\{\{1,2\},2,\heartsuit\}\}\\
\mathcal{P}(B) &= \{\emptyset,\{1\},\{2\},\{1,2\}\}\\
\mathcal{P}(A\cap B) &= \{\emptyset,\{2\}\}\\
\mathcal{P}(A\cap \{B\}) &= \{\emptyset,\{\{1,2\}\}\}
\end{aligned}$$

A.1.5

(a) $-2,-1,0,1,2$

(b) $(0,7),(1,4),(2,1)$

(c) $(1,-8),(2,-4),(4,-2),(8,-1)$

A.2.9 $a_1 = 2$, $a_2 = 3$

A.2.10 $a_n = F_{n+2}$

A.2.11 $a_n = \dfrac{n^2}{2}$

A.2.14 $a_n = (An + B)2^n$, where A and B are arbitrary constants.

A.2.15 $a_n = 3 + (-1)^n - n$

A.2.16 $a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ F_{n/2+1} & \text{if } n \text{ is even.} \end{cases}$

A.2.17 $\frac{\sqrt{3}+2}{2\sqrt{3}} \cdot \left(1+\sqrt{3}\right)^n + \frac{\sqrt{3}-2}{2\sqrt{3}} \cdot \left(1-\sqrt{3}\right)^n$

A.2.19 $a_n = (c+n) \cdot 3^{n-1} - \frac{2n+1}{4}$, where c is an arbitrary constant.

A.2.20 $a_n = \frac{2^{n+2} + (-1)^{n+1} - 3}{6} - n$

A.3.1 New: $23^3 \cdot 10^2 \cdot 22\,(= 26767400)$. Old: $23^3 \cdot 10^3\,(= 12167000)$.

A.3.2 $23 \cdot 22 \cdot 21 \cdot 10 \cdot 9 \cdot 8\,(= 7650720)$

A.3.3 $\binom{m}{2}$

A.3.4 $8!\,(= 40320)$

A.3.5 $7 \cdot 8!\,(= 282240)$

A.3.6 $8!\,(= 40320)$

A.3.7 $(m-1)!m!$

A.3.8 370

A.3.9 $\frac{109}{\binom{47}{5}}$ (≈ 0.000071)

A.3.10 $\binom{8}{2} \cdot 7!$ $(= 141120)$

A.3.11 $\binom{n+2}{3} \cdot n! + \frac{1}{2}\binom{n+2}{2}\binom{n}{2} \cdot n! = \frac{n(3n+1) \cdot (n+2)!}{24}$

A.3.15 $\binom{9}{3,2,2,1,1} = 3 \cdot 7!\,(= 15120)$

A.3.16 $\binom{6}{2,2,1,1} \cdot \binom{7}{3} = 6300$

A.3.17 $\binom{15}{3,3,3,3,3}$ (= 168168000)

A.3.18 $\binom{17}{3} = 680$

A.3.19 $\binom{18}{4} = 3060$

A.3.20 $\binom{17}{3} = 680$

A.3.21 $\binom{n-4}{5}$

A.3.22 $\binom{n-1}{4}$

A.3.27 $n! - 2(n-1)! = (n-2)(n-1)!$

A.3.28 $n! - (2(n-1)! + 2(n-1)! - 2(n-2)!) = (n-2)(n-3)(n-2)!$

A.3.29 35 students passed.

A.3.30 $26! - 23! - 22! - 22! + 20! + 21!\,(\approx 4 \cdot 10^{26})$

A.3.31 1980

A.3.32 $\binom{18}{4} - 4\binom{12}{4} + \binom{4}{2}\binom{6}{4} = 1170$

A.4.1 $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

A.4.2 $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$

A.4.5 $\gcd(693, 990) = 99$, $\operatorname{lcm}(693, 990) = 6930$

A.4.6 $2(k_1+1)(k_2+1)\cdots(k_m+1)$ divisors, $(k_1+1)(k_2+1)\cdots(k_m+1)$ positive divisors

A.4.7 567

A.4.8 6

A.4.10 $\alpha = 9$, $\beta = -19$, for example

A.4.11 $x = -17 + 74k$, $y = 8 - 35k$, $k \in \mathbb{Z}$

A.4.12 There are no solutions.

A.4.13 $x = 8$, $y = -29$ is the only one.

A.4.14 $x = 2 + 7k$, $y = -1 + 2\ell$, $z = -5k - \ell$, $k, \ell \in \mathbb{Z}$

A.4.15 $x = \pm 7$, $y = \pm 3$

A.5.1 (i) yes (ii) no (iii) no (iv) no

A.5.2 (i) no (ii) no (iii) yes (iv) yes

A.5.3 (i) $2^6 = 64$ (ii) $2^6 = 64$ (iii) $2^3 \cdot 3^3 = 216$

A.5.4 An equivalence class consists of all socks of any fixed colour that occurs among your (single coloured) socks.

A.5.5 The equivalence classes are the circles with 0 as centre in the complex plane (including the radius zero "circle" which only contains 0 itself).

A.5.6 An equivalence class is the set of all subsets of $\{1, 2, \ldots, 9\}$ that have a fixed number (between 0 and 9) of elements. $|[\{1, 2, 3\}]| = \binom{9}{3} = 84$.

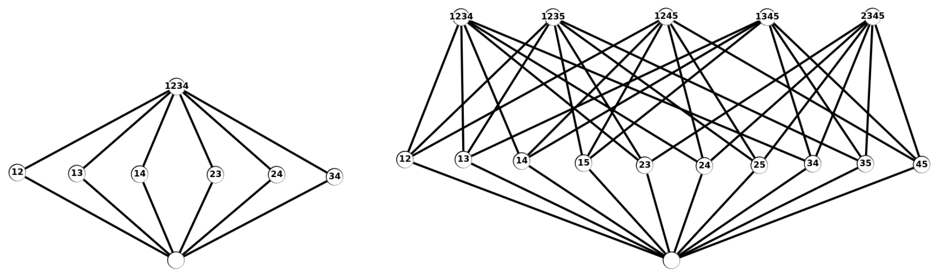
A.5.7 $[3] = \{3, 12, 27, 48, \ldots\}$

A.5.9 The equivalence classes are the lines that are parallel to ℓ.

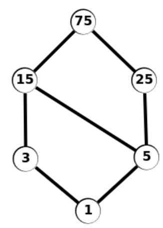
A.5.11 1

A.5.12 The empty set $\emptyset$ is the only minimal element. If $|X|$ is even, X is the only maximal element. If $|X|$ is odd, the sets of the form $X \setminus \{x\}$ for $x \in X$ are the maximal elements. Thus, $\mathcal{P}_{\text{even}}(X)$ always has a minimum. It has a maximum if and only if $|X|$ is even or $|X| = 1$.

A.5.13



A.5.14



A.5.17 The Hasse diagram of $P \oplus Q$ is constructed by placing the diagram of Q above the diagram of P and connecting all minimal Q-elements with all maximal P-elements.

A.5.18 One of many examples is $\mathbb{Z}$ ordered by $\leq$.

A.6.1 8

A.6.2 No, $3^4 \not\equiv 1$, for example. 3 is not invertible modulo 15.

A.6.3 12

A.6.5 4, 9

A.6.7 14 is invertible. The inverse is 11.

A.6.8 $x = 26 + 51k$, $k \in \mathbb{Z}$

A.6.9 115

A.6.10 $x = 105 + 11760k$, $k \in \mathbb{Z}$

A.6.11 $x = -4 + 30k$, $k \in \mathbb{Z}$

A.6.12 14

A.6.13 The same weekday that it was the day before yesterday.

A.6.14 10

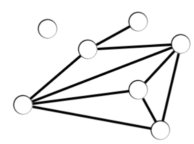
A.6.15 $x = 3 + 23k$, $k \in \mathbb{Z}$

A.6.17 40

A.6.18 2

A.6.19 63

A.7.1 Neither (a) nor (c) exist. (b):



A.7.2 $\binom{n}{2}$

A.7.3 $\frac{nk}{2}$ (Why is this an integer?)

A.7.4 $\frac{2\epsilon}{k}$

A.7.6 3

A.7.7 They are not isomorphic. (Many properties that are preserved by isomorphisms differ. For example, only the left graph contains a 3-cycle.)

A.7.8 One isomorphism is $1 \mapsto 10$, $2 \mapsto 11$, $3 \mapsto 7$, $4 \mapsto 8$, $5 \mapsto 9$, $6 \mapsto 12$.

A.7.9 5

A.7.10 The graphs in P.7.7 both contain eulerian paths but not eulerian circuits. The graphs in P.7.8 contain neither. The rightmost graph in P.7.7 contains a hamiltonian path, but no hamiltonian circuit. The other graphs contain hamiltonian circuits (hence also hamiltonian paths).

A.7.12 It suffices to add one edge, namely $\{1, 3\}$.

A.7.13 For even n. (If you consider the vertices as points in $\mathbb{R}^n$ with coordinates 0 or 1, then they are the corners of the n-dimensional unit cube, and the edges of the graph become, well, the edges of that cube.)

A.7.16 3-cycles: $\binom{n}{3}$, 4-cycles: $\binom{n}{4} \cdot 3$

A.7.17 The rightmost graph in P.7.7 is the only one which is bipartite.

A.7.18 4-cycles: $\binom{m}{2}\binom{n}{2}$, 6-cycles: $\binom{m}{3}\binom{n}{3} \cdot 6$

A.7.20 15

A.7.21 10

A.7.22 $\binom{n-1}{2}$

A.7.23 $2^{\frac{(n-1)(n-2)}{2}}$

A.7.25 Necessarily eight.

A.7.26 Two or three.

A.7.34 Left: $\chi(G) = 3$, $P(G, x) = x(x-1)(x-2)(x^2 - 3x + 3)$.
Right: $\chi(G) = 2$, $P(G, x) = x(x-1)(x^3 - 5x^2 + 10x - 7)$.

A.7.35 $x^m(x-1)^{n-m}$

A.7.39 $P(W_n, x) = x(x-2)^n + (-1)^n x(x-2)$

Walkthroughs

W.2.5 Note that $12 = 3+3+3+3$, $13 = 7+3+3$, and $14 = 7+7$. Fix an integer $n \geq 14$ and assume, in order to use strong induction, that k is a sum of 3's and 7's for all integers k satisfying $12 \leq k \leq n$. We want to prove that $n+1$ is also a sum of 3's and 7's. Since $12 \leq n-2 \leq n$, $n-2$ is a sum of 3's and 7's by the induction assumption. Hence, so is $n+1 = (n-2)+3$.

W.2.7 In the base case $n = 0$, there are no lines and just one region. The assertion is true in this case since $\dfrac{0(0+1)}{2} + 1 = 1$.

In order to use induction, fix $k \in \mathbb{N}$ and assume that every drawing of k straight lines cuts the plane into at most $\dfrac{k(k+1)}{2} + 1$ regions. Consider now a drawing of $k+1$ lines. We need to prove that the number of regions is at most $\dfrac{(k+1)(k+2)}{2} + 1$. Let L be the last of the $k+1$ lines that we draw. Before L is drawn, there are at most $\dfrac{k(k+1)}{2} + 1$ regions by the induction assumption. The line L splits every such region that it passes through into two new regions. Since L intersects other lines in at most k points, at most $k+1$ old regions are split into two new in this way. Therefore, the number of regions after L is drawn is at most $\dfrac{k(k+1)}{2} + 1 + k + 1 = \dfrac{(k+1)(k+2)}{2} + 1$, as desired.

W.2.9 Call a set *valid* if it does not contain two successive integers. Suppose $n \geq 3$ is an integer. Some of the valid subsets of $\{1, \ldots, n\}$ contain n and some do not. Those that do not are precisely the valid subsets of $\{1, \ldots, n-1\}$, and there are a_{n-1} of them. Those that do contain n are precisely the valid subsets of $\{1, \ldots, n-2\}$ with n appended. Hence, there are a_{n-2} of them. Summing up, we obtain $a_n = a_{n-1} + a_{n-2}$.

W.2.18 If $n \in \mathbb{Z}_+$, $\displaystyle\sum_{j=0}^{n} j^2 = \sum_{j=0}^{n-1} j^2 + n^2$. Thus, $a_n = a_{n-1} + n^2$ for such n. The homogeneous part of the general solution to this recurrence equation is

$$a_n^{\text{hom}} = A \cdot 1^n = A,$$

an arbitrary constant. To find a particular solution, we make the ansatz

$$a_n^{\text{part}} = Bn^3 + Cn^2 + Dn.$$

(Naïvely trying a polynomial of degree 2 won't work because its constant term is a homogeneous solution.) Plugging it into the recurrence equation, we get

$$Bn^3 + Cn^2 + Dn = B(n-1)^3 + C(n-1)^2 + D(n-1) + n^2,$$

which (as is seen after some shuffling) is equivalent to

$$(3B-1)n^2 + (-3B+2C)n + B - C + D = 0,$$

which has as only solution $B = \frac{1}{3}$, $C = \frac{1}{2}$, $D = \frac{1}{6}$. Thus,

$$a_n = A + \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = A + \frac{n(n+1)(2n+1)}{6}.$$

Using that $a_0 = 0$, we conclude $A = 0$ and arrive at the formula we hoped for.

W.3.10 Construct such a function by first choosing the two elements in the domain that should have the same function value; there are $\binom{8}{2}$ possibilities. After that, choose their common value (7 possibilities). Finally, choose the unique values for the remaining six elements in the domain (6! possibilities). Hence, there are $\binom{8}{2} \cdot 7!$ such functions.

W.3.21 In such a subset, think of the five selected elements as bars and the $n-5$ not selected elements as stars. (For example, if $n = 13$, "$**|*|***|*|*|$" would represent $\{3, 5, 9, 11, 13\}$).

Construct such a sequence by starting with a sequence of $n-5$ stars and then selecting five of the spaces between (or before, or after) the stars; the selected spaces are where the bars are placed. There are $n-4$ spaces, hence $\binom{n-4}{5}$ sequences of this form.

W.3.23 For notational convenience, let us say that $S_0 = \emptyset$ and $S_{k+1} = \{1, 2, \ldots, n\}$. A k-tuple of the described form is determined by choosing, for each integer j, $1 \le j \le n$, which of the sets $S_1, \ldots, S_{k+1}$ is the *first* that j appears in. (In other words, choosing the m for which it holds that $j \notin S_{m-1}$ and $j \in S_m$.) Since there are $k+1$ choices for each such j, the desired assertion follows.

W.3.26 Cut the range of arctan, i.e. the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, into six subintervals of equal length:

$$\begin{aligned}
I_1 &= \left]-\frac{\pi}{2}, -\frac{\pi}{3}\right], \\
I_2 &= \left]-\frac{\pi}{3}, -\frac{\pi}{6}\right], \\
I_3 &= \left]-\frac{\pi}{6}, 0\right], \\
I_4 &= \left]0, \frac{\pi}{6}\right], \\
I_5 &= \left]\frac{\pi}{6}, \frac{\pi}{3}\right], \\
I_6 &= \left]\frac{\pi}{3}, \frac{\pi}{2}\right[.
\end{aligned}$$

By the pigeonhole principle, at least two of the seven numbers, x_1 and x_2, have arctan values in the same interval I_j. Assign the names so that $x_1 \ge x_2$. Then, $0 \le \arctan x_1 - \arctan x_2$. Since the length of I_j is $\frac{\pi}{6}$, $\arctan x_1 - \arctan x_2 < \frac{\pi}{6}$. (The inequality is strict since I_j does not contain both endpoints.)

W.3.32 Let $\mathcal{U}$ denote the set of *all* nonnegative integer solutions (x_1, x_2, x_3, x_4) to the given inequality. For $i \in \{1, 2, 3, 4\}$, let $A_i \subseteq \mathcal{U}$ be the subset which consists of those solutions that satisfy $x_i \geq 6$. We seek $|\mathcal{U} \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)|$. Aiming to apply PIE, we consider intersections of the sets A_i. Note that any intersection of three (or all four) of them is empty, since if three variables exceed 5, their sum will definitely exceed 14. Moreover, the symmetry of the restrictions on the variables implies

$$|A_1| = |A_2| = |A_3| = |A_4|$$

and

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4|.$$

It therefore follows by PIE that

$$|\mathcal{U} \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)| = |\mathcal{U}| - 4|A_1| + 6|A_1 \cap A_2|.$$

Considering the slack variable $x_5 = 14 - (x_1 + x_2 + x_3 + x_4)$ yields that $|\mathcal{U}|$ is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 14$. By a "stars and bars"-argument (or a known formula) this number is $\binom{14+4}{4}$.

Introducing $y_1 = x_1 - 6$, we find that $|A_1|$ is the number of nonnegative integer solutions to $y_1 + x_2 + x_3 + x_4 + x_5 = 8$. Thus, $|A_1| = \binom{8+4}{4}$.

Finally, setting $y_2 = x_2 - 6$, $|A_1 \cap A_2|$ is the number of nonnegative integer solutions to $y_1 + y_2 + x_3 + x_4 + x_5 = 2$, namely $\binom{2+4}{4}$.

Hence, the given inequality has $\binom{18}{4} - 4\binom{12}{4} + 6\binom{6}{4} = 1170$ solutions of the specified form.

W.4.7 For $k \in \mathbb{Z}_+$, define $A_k = \{N \in \{1, \ldots, 2000\} : k|N\}$. We wish to compute $|A_6 \cup A_8 \cup A_{15}|$. By the principle of inclusion-exclusion, this number is equal to

$$|A_6| + |A_8| + |A_{15}| - |A_6 \cap A_8| - |A_6 \cap A_{15}| - |A_8 \cap A_{15}| + |A_6 \cap A_8 \cap A_{15}|.$$

Several integers simultaneously divide N if and only if their least common multiple does. Hence,

$$\begin{aligned}
|A_6 \cup A_8 \cup A_{15}| &= |A_6| + |A_8| + |A_{15}| - |A_{24}| - |A_{30}| - |A_{120}| + |A_{120}| \\
&= \left\lfloor \frac{2000}{6} \right\rfloor + \left\lfloor \frac{2000}{8} \right\rfloor + \left\lfloor \frac{2000}{15} \right\rfloor - \left\lfloor \frac{2000}{24} \right\rfloor - \left\lfloor \frac{2000}{30} \right\rfloor \\
&= \frac{1998}{6} + \frac{2000}{8} + \frac{399}{3} - \frac{249}{3} - \frac{198}{3} \\
&= 333 + 250 + 133 - 83 - 66 \\
&= 567.
\end{aligned}$$

(Here, $\lfloor \cdot \rfloor$ denotes the *floor function* that rounds it input down to the nearest weakly smaller integer.)

W.4.9 We use Euclid and compute

$$\begin{aligned} 3n^2 + n - 7 &= 3 \cdot (n^2 - 2) + n - 1, \\ n^2 - 2 &= n \cdot (n - 1) + n - 2, \\ n - 1 &= 1 \cdot (n - 2) + 1. \end{aligned}$$

Hence, $\gcd(3n^2 + n - 7, n^2 - 2) = \gcd(n^2 - 2, n - 1) = \gcd(n - 1, n - 2) = \gcd(n - 2, 1) = 1$, as desired. $\square$

(*Remark.* Technically speaking, if $n \in \{0, 1, 2, 3\}$, we are not quite using the division algorithm for integers in the usual fashion here, because not all the "remainders" $n - 1$, $n - 2$, and 1 are both nonnegative and smaller than the corresponding denominators. However, the *equations* are still true, and they still imply the conclusion. The purist could consider those four cases separately.)

W.5.7 The relation $\bowtie$ is reflexive since xx is a square for all $x \in \mathbb{Z}_+$. It is symmetric since $xy = n^2 \Leftrightarrow yx = n^2$. Finally, it is transitive because if $xy = m^2$ and $yz = n^2$ for $m, n \in \mathbb{Z}$, then $xz = \left(\frac{mn}{y}\right)^2$, and $\frac{mn}{y} \in \mathbb{Z}$ since $xz \in \mathbb{Z}$. $\square$

We have $[3] = \{3m^2 : m \in \mathbb{Z}_+\} = \{3 \cdot 1^2, 3 \cdot 2^2, 3 \cdot 3^2, 3 \cdot 4^2, \ldots\}$.

W.5.15 Both -1 and 1 are divisors of n for every $n \in \mathbb{N}$. Since $-1|1$ and $1|-1$, but $-1 \neq 1$, the divisibility relation is not antisymmetric.

W.5.20 Pick $a, b \in D_n$. Then, $\gcd(a, b)$ divides a and b (hence also n). Moreover, if c divides a and b, then c divides $\gcd(a, b)$. Hence $\gcd(a, b)$ is the meet (= unique greatest lower bound) of a and b in D_n. Completely analogously, one verifies that $\mathrm{lcm}(a, b)$ is the join (= unique least upper bound) of a and b in D_n.

W.6.6 We compute all possible values modulo 7:

$$\begin{aligned} &0^3 - 3 \cdot 0 + 1 = 1 \not\equiv 0, \\ &1^3 - 3 \cdot 1 + 1 = -1 \equiv 6 \not\equiv 0, \\ &2^3 - 3 \cdot 2 + 1 = 3 \not\equiv 0, \\ &3^3 - 3 \cdot 3 + 1 = 19 \equiv 5 \not\equiv 0, \\ &4^3 - 3 \cdot 4 + 1 = 53 \equiv 4 \not\equiv 0, \\ &5^3 - 3 \cdot 5 + 1 \equiv -8 + 6 + 1 \equiv 6 \not\equiv 0, \\ &6^3 - 3 \cdot 6 + 1 \equiv -1 + 3 + 1 = 3 \not\equiv 0. \quad \square \end{aligned}$$

W.6.11 For an integer x, $p(x) = x^4 + 2x - 8$ is divisible by 60 if and only if $p(x)$ is divisible by 3, 4, and 5. We compute all values modulo these three moduli:

$$\begin{aligned}
p(0) &= -8 \not\equiv 0 \pmod 3,\\
p(1) &= -5 \not\equiv 0 \pmod 3,\\
p(2) &= 12 \equiv 0 \pmod 3,\\
p(0) &= -8 \equiv 0 \pmod 4,\\
p(1) &= -5 \not\equiv 0 \pmod 4,\\
p(2) &= 12 \equiv 0 \pmod 4,\\
p(3) &= 79 \not\equiv 0 \pmod 4,\\
p(0) &= -8 \not\equiv 0 \pmod 5,\\
p(1) &= -5 \equiv 0 \pmod 5,\\
p(2) &= 12 \not\equiv 0 \pmod 5,\\
p(3) &= 79 \not\equiv 0 \pmod 5,\\
p(4) &= 256 \not\equiv 0 \pmod 5.
\end{aligned}$$

Hence, we want all solutions to the system of congruences $x \equiv 2 \pmod 3$, $x \equiv 0$ or $2 \pmod 4$, and $x \equiv 1 \pmod 5$. The Chinese remainder theorem tells us that the solutions modulo 60 are

$$x = 2 \cdot (-1) \cdot 4 \cdot 5 + (0 \text{ or } 2) \cdot (-1) \cdot 3 \cdot 5 + 1 \cdot 3 \cdot 3 \cdot 4 = (-4 \text{ or } -34).$$

Finally, we observe that $x \equiv (-4 \text{ or } -34) \pmod{60} \Leftrightarrow x = -4 + 30k$, $k \in \mathbb{Z}$.

(*Remark.* We could have cut some corners by noticing that $x \equiv (0 \text{ or } 2) \pmod 4$ is in fact equivalent to $x \equiv 0 \pmod 2$. We could then have applied CRT for the solutions modulo 30 directly.)

W.6.19 Since $91 = 7 \cdot 13$, the private key d is the inverse of 29 modulo $6 \cdot 12 = 72$. Using Euclid (or trial-and-error), we find $5 \cdot 29 = 2 \cdot 72 + 1$. Hence, $d = 5$. Thus, the plaintext is $7^5 = 343 \cdot 49 \equiv -21 \cdot 49 = -3 \cdot 343 \equiv -3 \cdot (-21) = 63 \pmod{91}$.

W.7.6 Since the graph is connected, no vertex has degree 0. Therefore, the degrees of the first five vertices are $1, 2, 3, 4, 5$. Let x_i denote the vertex of degree i for $i \in \{1, 2, 3, 4, 5\}$, and let y denote the sixth vertex. Since x_5 is connected to all vertices, x_1 is only connected to x_5. Hence, x_4 is connected to every vertex except x_1. Thus, x_2 is only connected to x_4 and x_5. Therefore, x_3 is connected to every vertex except x_1 and x_2. Thus, y is connected to x_3, x_4, and x_5, but neither to x_1 nor to x_2. We conclude that the degree of y is 3.

W.7.14 Suppose G has $2m$ vertices and let d denote their common degree. Since K_{2m} is $(2m-1)$-regular, $\overline{G}$ is $(2m-1-d)$-regular. If $d \geq m$, G is hamiltonian by Dirac's theorem and we are done. If not, $d \leq m-1$. This implies that $2m - 1 - d \geq 2m - 1 - (m-1) = m$, so that, again by Dirac's theorem, $\overline{G}$ is hamiltonian.

W.7.19 Let G be the graph described in the statement of the problem. Consider a cycle in G. It is a sequence of regions $R_1, R_2, \ldots, R_m, R_{m+1} = R_1$ where R_i and R_{i+1} are separated by exactly one of the lines. Let L_i be the line separating R_i from R_{i+1}. Since R_i and R_{i+1}

are on different sides of L_i for every i, each line must appear an even number of times in the list $L_1, \dots, L_m$ since the walk comes back to R_1, i.e. to the same side of every line. Thus, m is even.

W.7.28 Suppose, in order to obtain a contradiction, that G is planar. We may assume G is connected (otherwise, just consider one connected component of G). Let v, e, and f be the number of vertices, edges and regions, respectively, of a planar embedding of G. Every region is incident to at least three edges, and every edge is incident to at most two regions. Hence, counting edge-region incidences, we obtain $3f \leq 2e$. By the handshake lemma, and the fact that every vertex degree is at least 6, $6v \leq 2e$. Thus, Euler's formula gives

$$2 = v - e + f \leq \frac{e}{3} - e + \frac{2e}{3} = 0,$$

which is the contradiction we wanted.

W.7.32 Let $C = \{1, 2, \dots, k+1\}$ be our set of colours. Order the vertices $v_1, v_2, \dots, v_n$. Let us colour them one by one in this order. When it is time to colour v_i, assign to it the smallest colour in C which has not already been assigned to one of the neighbours of v_i. Since the degree of every vertex is less than $|C|$, there is always a colour available.

W.7.38 Let $x \in \mathbb{Z}_+$. A proper x-colouring of $\mathcal{S}(G)$ either assigns the same colour to v_1 and v_2, or they get different colours. A colouring where they receive the same colour can be constructed by first choosing their common colour (x choices) and then choosing a proper colouring of G using the remaining $x-1$ colours ($P(G, x-1)$ possibilities). Thus, $xP(G, x-1)$ is the number of colourings where v_1 and v_2 have the same colour. A colouring where they have different colours can be produced by first choosing the colour of v_1 (x possibilities), then that of v_2 ($x-1$ possibilities) and finally a colouring of G using the remaing $x-2$ colours. Hence, there are $x(x-1)P(G, x-2)$ colourings of this form. Adding the two yields the desired total number of proper x-colourings of $\mathcal{S}(G)$.