Chromatic polynomials

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Let $G = (V, E)$ be a (finite, simple, undirected) graph and $k \in \mathbb{N}$. Recall that a *proper k-colouring* of G is a function $f : V \to \{1, 2, \ldots, k\}$ such that $\{v_1, v_2\} \in E \Rightarrow f(v_1) \neq f(v_2)$. In other words, a proper k-colouring is a colouring of the vertices of G that uses *some of* (maybe all, but not necessarily so) k specified colours such that every edge gets different colours on its vertices.

Somewhat surprisingly, the proper k-colourings of G are counted by a polynomial function.

Theorem 1. *Given G, there is a unique polynomial $P(G, x)$ such that $P(G, k)$ is equal to the number of proper k-colourings of G for every $k \in \mathbb{N}$.*

The uniqueness part of Theorem 1 is not deep, for if two polynomials have the same values on an infinite set of inputs, they must coincide.[1] Existence follows from Corollary 10 below.

Definition 2. *The* chromatic polynomial *of G is the polynomial $P(G, x)$ whose existence and uniqueness is guaranteed by Theorem 1.*

For some graphs G, $P(G, x)$ can be computed directly using the combinatorial multiplication principle by considering the vertices one at a time, sometimes in an appropriately chosen order.

Example 3. *We may construct any proper k-colouring of the complete graph K_n by first choosing the colour of vertex 1 (k choices), then the colour of vertex 2 ($k-1$) choices, and so on. In total, K_n has $k(k-1)(k-2)\cdots(k-n+1)$ proper k-colourings. Hence, the chromatic polynomial of K_n is $P(K_n, x) = x(x-1)(x-2)\cdots(x-n+1)$.*

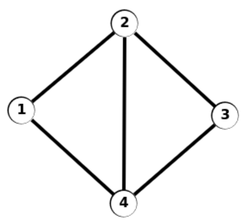


Figure 1: Illustration for Example 4.

Example 4. *Consider the graph G depicted in Figure 1. If we try to count its proper k-colourings by colouring the vertices in numerical order, as in Example 3, we run into trouble when reaching vertex 4, since the number of choices we have for its colour depends on whether the vertices 1 and 3 have received different colours or not. We could look at the two different cases separately. However, an easier way out is to consider the vertices in the order 1, 2, 4, 3. Then, we have k choices for vertex 1, $k-1$ for 2, $k-2$ for 4, and $k-2$ also for 3. Hence, $P(G, x) = x(x-1)(x-2)^2$.*

[1] Proof: The difference between the two polynomials will have an infinite number of roots. By the factor theorem, that can only happen if this difference is the zero polynomial.

Remark 5. *What made it possible to argue directly "one vertex at a time" in Examples 3 and 4 was that it was possible to order the vertices of G in such a way that for every vertex v, those neighbours of v that came before v in this order formed a complete subgraph of G (i.e., they were all neighbours of each other). This allowed us to conclude that the colours chosen for those neighbours were all different. Graphs that admit such a vertex order are called* chordal. *Alas, many graphs are not chordal.*

Observe that the chromatic polynomial $P(G, x)$ contains much more information than the chromatic number $\chi(G)$. Indeed, it follows immediately from the definitions that $\chi(G)$ is the smallest nonnegative integer which is not a root of $P(G, x)$. For example, with G as in Figure 1, the result of Example 4 shows $P(G, 0) = P(G, 1) = P(G, 2) = 0$ but $P(G, 3) \neq 0$. Hence, $\chi(G) = 3$. This just tells us that it is possible to properly colour G using 3 (but not 2) colours. More informatively, $P(G, 3) = 3 \cdot 2 \cdot 1^2 = 6$ reveals that G has precisely six different proper 3-colourings.

Remark 6. *It is perfectly possible for (non-chordal) graphs to have chromatic polynomials with some non-integer roots. Such roots bear no obvious relation to colourings. Example 11 below, for example, will reveal that the chromatic polynomial of the graph shown in Figure 3 has an irrational, real root as well as two complex, non-real roots.*

Our next goal is to develop a recursive formula for the chromatic polynomial of (almost) any graph in terms of the chromatic polynomials of smaller graphs. This formula is the key to most properties of chromatic polynomials; in particular it will help us attack non-chordal graphs.

Definition 7. *Let $G = (V, E)$ be a graph with $E \neq \emptyset$ and let $e = \{v_1, v_2\} \in E$ be any edge. We define two new graphs as follows:*

- *The* deletion *of e is the graph $G - e$ obtained from G by removing the edge e. That is,*

$$G - e = (V, E \setminus \{e\}).$$

- *The* contraction *of e is the graph $G \cdot e$ obtained from G by identifying v_2 with v_1 (and removing e and any double edges that appear in the process).*

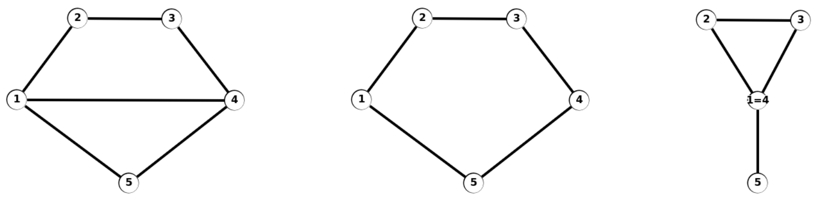


Figure 2: Deletion and contraction of the edge $\{1, 4\}$.

Example 8. *If G is the leftmost graph in Figure 2 and $e = \{1, 4\}$, then $G - e$ is the graph in the middle and $G \cdot e$ is the one on the right.*

The point of all this is the following theorem which provides a recursive formula for the chromatic polynomial of any graph that has at least one edge.

Theorem 9 (Deletion-contraction). *If e is an edge of the graph G, then*

$$P(G, x) = P(G - e, x) - P(G \cdot e, x).$$

Proof. A proper k-colouring of $G - e$ either gives the same colour to the vertices of e, or else they receive different colours. The colourings of the former kind are precisely the proper k-colourings of $G \cdot e$, whereas those of the latter kind are the proper k-colourings of G. Hence, $P(G - e, k) = P(G \cdot e, k) + P(G, k)$. $\square$

An immediate corollary is that chromatic polynomials are well-defined, proving Theorem 1.

Corollary 10. *The number of proper k-colourings of G is a polynomial function of k.*

Proof. We induct on the number of edges of G. In the base case, when there are n vertices and no edges, the number of k-colourings of G is k^n which is a polynomial in k. The induction step is provided by Theorem 9, since the difference of two polynomials is a polynomial. (Note that $G - e$ and $G \cdot e$ have strictly fewer edges than G, so we may apply the (strong) induction assumption to them.) $\square$

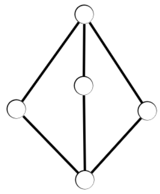
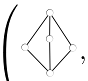
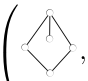
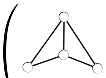
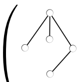
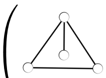
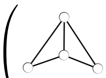


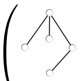
Figure 3: Illustration for Example 11.

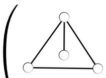
Example 11. *We compute the chromatic polynomial of the graph in Figure 3 by repeated application of Theorem 9:*

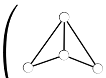
$P\Bigg($  $, x\Bigg) = P\Bigg($  $, x\Bigg) - P\Bigg($  $, x\Bigg)$

$= \Bigg(P\Bigg($  $, x\Bigg) - P\Bigg($  $, x\Bigg)\Bigg) - P\Bigg($  $, x\Bigg).$

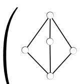
We are left with the task of computing chromatic polynomials of three smaller graphs. They are all chordal, hence can be coloured one vertex at a time in the spirit of Examples 3 and 4. This yields

$P\Bigg($  $, x\Bigg) = x(x-1)^4,$

$P\Bigg($  $, x\Bigg) = x(x-1)^2(x-2),$

$P\Bigg($  $, x\Bigg) = x(x-1)(x-2)^2,$

as the reader should pause to convince her/himself of. Therefore,

$P\Bigg($  $, x\Bigg) = x(x-1)^4 - x(x-1)^2(x-2) - x(x-1)(x-2)^2$

$$\begin{aligned}
&= x(x-1)\left((x-1)^3 - (x-1)(x-2) - (x-2)^2\right) \\
&= x(x-1)(x^3 - 5x^2 + 10x - 7).
\end{aligned}$$

Example 12. *It is sometimes convenient to use Theorem 9 "backwards". For example, let G_n be the graph obtained from the complete graph K_n by removing one edge.*[a] *Let us compute its chromatic polynomial.*
Since $G_n = K_n - e$, for some edge e, Theorem 9 yields

$$\begin{aligned} P(G_n, x) &= P(K_n, x) + P(K_n \cdot e, x) \\ &= P(K_n, x) + P(K_{n-1}, x) \\ &= x(x-1)\cdots(x-n+1) + x(x-1)\cdots(x-n+2) \\ &= x(x-1)\cdots(x-n+2)(x-n+1+1) \\ &= x(x-1)\cdots(x-n+3)(x-n+2)^2. \end{aligned}$$

This result could also have been obtained by a direct vertex-by-vertex argument, because G_n is chordal. Note also that the case $n = 4$ appeared already in Example 4.

[a] It does not matter which edge since all such graphs are isomorphic, hence have the same chromatic polynomial.