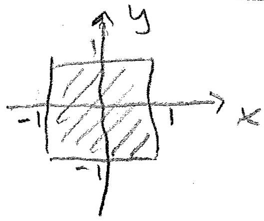
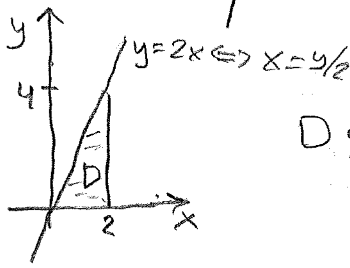
Bilder på mängder, tips 6.2a, 6.3, 6.4a-d, 6.5ac, 6.6, 6.8a

Lösningar därefter

6.2a $\begin{cases} |x| \le 1 \\ |y| \le 1 \end{cases} \Leftrightarrow \begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases}$

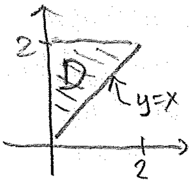


6.3 Bild på $\underbrace{0 \le y \le 2x \le 4}_{D}$



D ges av $\begin{cases} 0 \le x \le 2 \\ 0 \le y \le 2x \end{cases}$ eller $\begin{cases} 0 \le y \le 4 \\ \frac{y}{2} \le x \le 2 \end{cases}$

6.4a

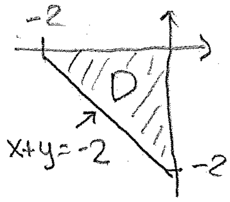


D: $\begin{cases} 0 \le x \le 2 \\ x \le y \le 2 \end{cases}$ eller $\begin{cases} 0 \le y \le 2 \\ 0 \le x \le y \end{cases}$

Partiell integration $\int (x-y)e^{x+y}\,dx = (x-y)e^{x+y} - \int 1 \cdot e^{x+y}\,dx$

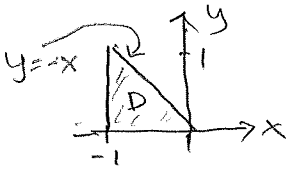
"y konstant"
primitiv på e^{x+y}, derivera $x-y$ (på x)

6.4b



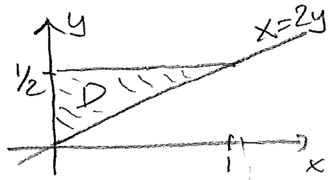
$\begin{cases} -2 \le x \le 0 \\ -2-x \le y \le 0 \end{cases}$ eller $\begin{cases} -2 \le y \le 0 \\ -2-y \le x \le 0 \end{cases}$

6.4c Primitiv av e^{x^2} på x omöjlig, börja integrera på y



$\begin{cases} -1 \le x \le 0 \\ 0 \le y \le -x \end{cases}$

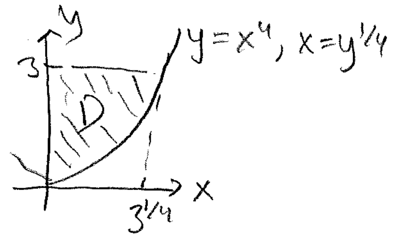
6.4d Börja integrera x



$\begin{cases} 0 \le y \le \frac{1}{2} \\ 0 \le x \le 2y \end{cases}$

(Forts. bilder, tips)

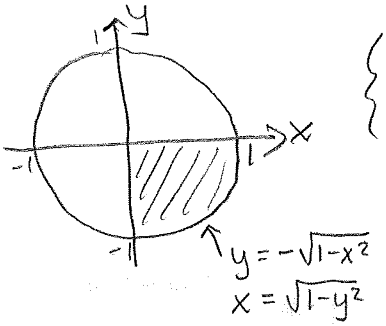
6.5a



$\begin{cases} 0 \le x \le 3^{1/4} \\ x^4 \le y \le 3 \end{cases}$ eller $\begin{cases} 0 \le y \le 3 \\ 0 \le x \le y^{1/4} \end{cases}$

Går bra börja med både x och y

6.5c

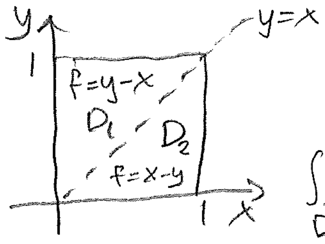


$\begin{cases} 0 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 0 \end{cases}$ eller $\begin{cases} -1 \le y \le 0 \\ 0 \le x \le \sqrt{1-y^2} \end{cases}$

$x^2+y^2=1 \Rightarrow y=\pm\sqrt{1-x^2}$ → + övre halvcirkeln, − undre

$\Rightarrow x=\pm\sqrt{1-y^2}$ → + högra halvcirkeln, − vänstra

6.6a $f(x,y)=|x-y|=\begin{cases} x-y & \text{då } y \le x \\ y-x & \text{då } y \ge x \end{cases}$

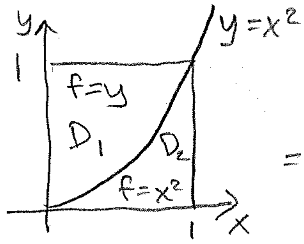


Dela upp i två integraler

$\iint_{D_1}(y-x)\,dxdy+\iint_{D_2}(x-y)\,dxdy$

6.6b $f(x,y)=\underbrace{\max(x^2,y)}_{}=\begin{cases} x^2 & \text{då } y \le x^2 \\ y & \text{då } y \ge x^2 \end{cases}$

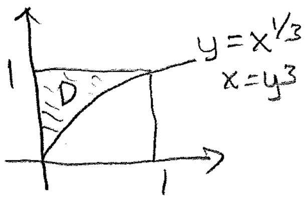
betyder att i varje punkt är f det största av x^2 och y



$\iint_D f(x,y)\,dxdy = \iint_{D_1} y\,dxdy + \iint_{D_2} x^2\,dxdy$

6.8a $\frac{1}{\sqrt{1+y^8}}$ kan vi ej hitta primitiv till → byt integrationsordning

Mängden $\begin{cases} 0 \le x \le 1 \\ x^{1/3} \le y \le 1 \end{cases}$ är



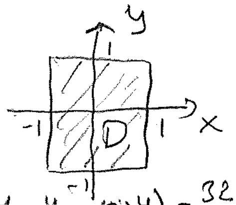
, kan skrivas $\begin{cases} 0 \le y \le 1 \\ 0 \le x \le y^3 \end{cases}$

ger $\int_0^1 \left(\int_0^{y^3} \frac{1}{\sqrt{1+y^8}}\,dx \right) dy$

$\int \frac{y^3}{\sqrt{1+y^8}}\,dy = ?$ Sätt $t=y^4$, kom ihåg $\int \frac{dt}{\sqrt{\alpha+t^2}} = \ln|t+\sqrt{\alpha+t^2}|\ (+C)$ från envariabeln

Lösningar

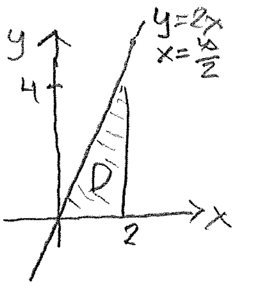
6.2a



$$\iint_D (x+y)^2\,dx\,dy = \int_{-1}^{1}\left(\int_{-1}^{1}(x+y)^2\,dx\right)dy =$$

$$= \int_{-1}^{1}\left[\frac{(x+y)^3}{3}\right]_{x=-1}^{1}dy = \int_{-1}^{1}\left(\frac{(1+y)^3}{3}-\frac{(-1+y)^3}{3}\right)dy = \left[\frac{(1+y)^4}{12}-\frac{(-1+y)^4}{12}\right]_{-1}^{1} = \frac{1}{12}\left(2^4-0^4-0^4+(-2)^4\right) = \frac{32}{12} = \frac{8}{3}$$

6.3 D: $0 \le y \le 2x \le 4$



D kan skrivas

$$\begin{cases} 0 \le x \le 2 \\ 0 \le y \le 2x \end{cases} \;(*) \iff \begin{cases} 0 \le y \le 4 \\ \frac{y}{2} \le x \le 2 \end{cases} \;(**)$$

$$I = \iint_D (xy+y^2)\,dx\,dy$$

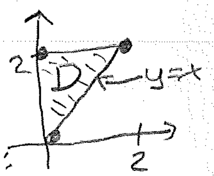
Med (*) fås $I = \int_0^2\left(\int_0^{2x}(xy+y^2)\,dy\right)dx$, med (**) $I = \int_0^4\left(\int_{y/2}^{2}(xy+y^2)\,dx\right)dy$

Uträknat från (*)

$$I = \int_0^2\left(\int_0^{2x}(xy+y^2)\,dy\right)dx = \int_0^2\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_{y=0}^{y=2x}dx = \int_0^2\left(x\cdot\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-x\cdot\frac{0^2}{2}-\frac{0^3}{3}\right)dx =$$

$$= \int_0^2 \underbrace{\left(2+\frac{8}{3}\right)}_{14/3}x^3\,dx = \frac{14}{3}\left[\frac{x^4}{4}\right]_0^2 = \frac{14}{3}\cdot\frac{16}{4} = \frac{56}{3}$$

6.4a $I = \iint_D (x-y)e^{x+y}\,dx\,dy$ D triangel med hörn i $(0,0),(0,2),(2,2)$



D kan skrivas $\begin{cases} 0 \le x \le 2 \\ x \le y \le 2 \end{cases}$ eller $\begin{cases} 0 \le y \le 2 \\ 0 \le x \le y \end{cases}$. Med andra varianten:

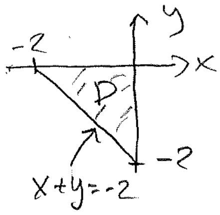
$$I = \int_0^2\left(\int_0^y (x-y)e^{x+y}\,dx\right)dy = \int_0^2\left(\left[(x-y)e^{x+y}\right]_{x=0}^{y} - \int_0^y 1\cdot e^{x+y}\,dx\right)dy =$$

partiell integration på x

$$= \int_0^2\left((y-y)e^{y+y}-(0-y)e^y-\left[e^{x+y}\right]_{x=0}^{y}\right)dy = \int_0^2\left(ye^y-e^{y+y}+e^{0+y}\right) = \left[\begin{array}{l}\int ye^y\,dy = \\ = ye^y-\int 1\cdot e^y\,dy = \\ = ye^y-e^y\end{array}\right] =$$

$$= \left[ye^y-\cancel{e^y}-\frac{1}{2}e^{2y}+\cancel{e^y}\right]_0^2 = 2e^2-\frac{1}{2}e^4-0+\frac{1}{2}e^0 = 2e^2-\frac{1}{2}e^4+\frac{1}{2}$$

6.4b $I=\iint_D (2+x+y)\,dxdy$ $\quad D: \begin{cases} x+y \geq -2 \\ x \leq 0 \\ y \leq 0 \end{cases}$

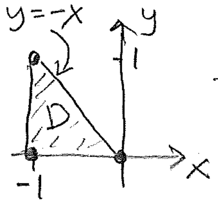


$\begin{cases} -2 \leq x \leq 0 \\ -2-x \leq y \leq 0 \end{cases}$ eller $\begin{cases} -2 \leq y \leq 0 \\ -2-y \leq x \leq 0 \end{cases}$

$$I=\int_{-2}^{0}\left(\int_{-2-x}^{0}(2+x+y)\,dy\right)dx=\int_{-2}^{0}\left[\tfrac{1}{2}(2+x+y)^2\right]_{y=-2-x}^{y=0}dx=\tfrac{1}{2}\int_{-2}^{0}\left((2+x+0)^2-(\underbrace{2+x-2-x}_{=0})^2\right)dx=$$

$$=\tfrac{1}{2}\left[\tfrac{1}{3}(2+x)^3\right]_{-2}^{0}=\tfrac{1}{6}(2^3-0^3)=\tfrac{4}{3}$$

6.4c $I=\iint_D e^{x^2}\,dxdy$

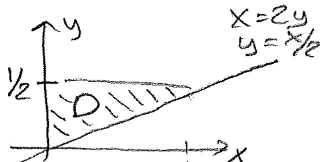


triangel $\begin{cases} -1 \leq x \leq 0 \\ 0 \leq y \leq -x \end{cases}$ (*) eller $\begin{cases} 0 \leq y \leq 1 \\ -1 \leq x \leq -y \end{cases}$

Primitiv på x ~ omöjlig, börja integrera y ⇒ (*) används

$$I=\int_{-1}^{0}\left(\int_{0}^{-x}e^{x^2}dy\right)dx=\int_{-1}^{0}\left[y\cdot e^{x^2}\right]_{y=0}^{y=-x}dx=\int_{-1}^{0}\left(-xe^{x^2}-0e^{x^2}\right)dx=\left[-\tfrac{1}{2}e^{x^2}\right]_{-1}^{0}=-\tfrac{1}{2}e^0+\tfrac{1}{2}e^1=\frac{e-1}{2}$$

6.4d $I=\iint_D \frac{x^3}{1+y^5}\,dxdy$ $\quad$, $D: 0 \leq x \leq 2y \leq 1$



$\begin{cases} 0 \leq x \leq 1 \\ \frac{x}{2} \leq y \leq \frac{1}{2} \end{cases}$ eller $\begin{cases} 0 \leq y \leq \frac{1}{2} \\ 0 \leq x \leq 2y \end{cases}$

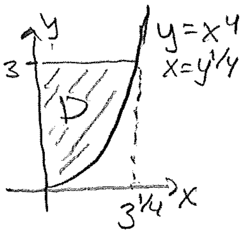
Börja integrera x ⇒

$$I=\int_{0}^{1/2}\left(\int_{0}^{2y}x^3dx\right)\frac{1}{1+y^5}dy=\int_{0}^{1/2}\left[\frac{x^4}{4}\right]_{0}^{2y}\frac{1}{1+y^5}dy=\int_{0}^{1/2}\frac{4y^4}{1+y^5}dy=\left[\tfrac{4}{5}\ln(1+y^5)\right]_{0}^{1/2}=$$

$$=\tfrac{4}{5}\left(\ln\left(1+\tfrac{1}{32}\right)-\underbrace{\ln 1}_{=0}\right)=\tfrac{4}{5}\ln\tfrac{33}{32}$$

$$\left[\int\frac{y^4}{1+y^5}dy=\Big/ \begin{matrix} t=y^5 \\ dt=5y^4dy \end{matrix} \Big/=\tfrac{1}{5}\int\frac{dt}{1+t}=\tfrac{1}{5}\ln(1+t)=\tfrac{1}{5}\ln(1+y^5)\right]$$

6.5a $I=\iint_D x^3y\,dxdy$ $\quad D: x^4 \leq y \leq 3, x \geq 0$



$\begin{cases} 0 \leq x \leq 3^{1/4} \\ x^4 \leq y \leq 3 \end{cases}$ eller $\begin{cases} 0 \leq y \leq 3 \\ 0 \leq x \leq y^{1/4} \end{cases}$

x först t.ex. ger

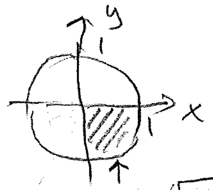
$$I=\int_{0}^{3}\left(\int_{0}^{y^{1/4}}x^3dx\right)y\,dy=\int_{0}^{3}\left[\frac{x^4}{4}\right]_{x=0}^{x=y^{1/4}}y\,dy=\tfrac{1}{4}\int_{0}^{3}(y-0)y\,dy=$$

$$=\tfrac{1}{4}\left[\frac{y^3}{3}\right]_{0}^{3}=\tfrac{1}{4}\cdot\frac{3^3}{3}=\frac{9}{4}$$

y först ger $I=\int_{0}^{3^{1/4}}x^3\left(\int_{x^4}^{3}y\,dy\right)dx=\int_{0}^{3^{1/4}}x^3\left[\frac{y^2}{2}\right]_{y=x^4}^{y=3}dx=\int_{0}^{3^{1/4}}x^3\left(\frac{9}{2}-\frac{x^8}{2}\right)dx=\frac{1}{2}\left[\frac{9x^4}{4}-\frac{x^{12}}{12}\right]_{0}^{3^{1/4}}=$

$$=\tfrac{1}{2}\left(\frac{27}{4}-\frac{27}{12}\right)=\frac{9}{4}$$

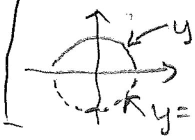
6.5c) $I=\iint_D xy\,dx\,dy$



$\begin{cases} 0\le x\le 1 \\ -\sqrt{1-x^2}\le y\le 0 \end{cases}$

$y=-\sqrt{1-x^2}$ (eller $x=\sqrt{1-y^2}$)

[OBS $x^2+y^2=1$
$\Rightarrow y=\pm\sqrt{1-x^2}$

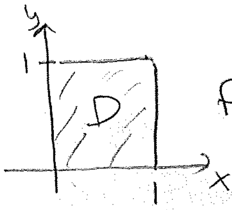


]

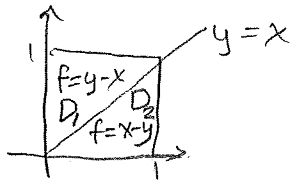
$$\Rightarrow I=\int_0^1 x\left(\int_{-\sqrt{1-x^2}}^{0} y\,dy\right)dx=\int_0^1 x\left[\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{0}dx=\int_0^1 x\left(0-\frac{1-x^2}{2}\right)dx=$$

$$=\frac{1}{2}\int_0^1(-x+x^3)\,dx=\frac{1}{2}\left[-\frac{x^2}{2}+\frac{x^4}{4}\right]_0^1=\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{4}\right)=-\frac{1}{8}$$

6.6a)



$f(x,y)=|x-y|=\begin{cases} x-y & \text{då } x\ge y \\ y-x & \text{då } y\ge x \end{cases}$



$D_1:\begin{cases} 0\le x\le 1 \\ x\le y\le 1 \end{cases}$

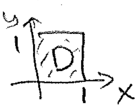
$D_2:\begin{cases} 0\le x\le 1 \\ 0\le y\le x \end{cases}$

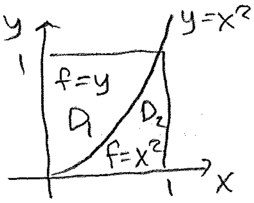
$$\Rightarrow \iint_D |x-y|\,dx\,dy=\iint_{D_1}(y-x)\,dx\,dy+\iint_{D_2}(x-y)\,dx\,dy=\int_0^1\left(\int_x^1(y-x)\,dy\right)dx+\int_0^1\left(\int_0^x(x-y)\,dy\right)dx=$$

$$=\int_0^1\left[\frac{(y-x)^2}{2}\right]_{y=x}^{1}dx+\int_0^1\left[-\frac{(x-y)^2}{2}\right]_{y=0}^{x}dx=\int_0^1\left(\frac{(1-x)^2}{2}-0\right)dx+\int_0^1\left(-0+\frac{x^2}{2}\right)dx=\left[-\frac{(1-x)^3}{6}+\frac{x^3}{6}\right]_0^1=0+\frac{1}{6}+\frac{1}{6}-0=\frac{1}{3}$$

[symmetri både i $f(x,y)=|x-y|$ och geometri i D ger att integraler över D_1 och D_2 blir lika]

6.6b) $f(x,y)=\max(x^2,y)=\begin{cases} x^2 & \text{då } y\le x^2 \\ y & \text{då } y\ge x^2 \end{cases}$





$D_1:\begin{cases} 0\le x\le 1 \\ x^2\le y\le 1 \end{cases}$

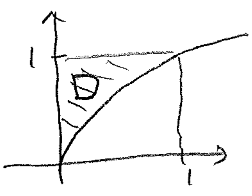
$D_2:\begin{cases} 0\le x\le 1 \\ 0\le y\le x^2 \end{cases}$

$$\iint_D f(x,y)\,dx\,dy=\iint_{D_1} y\,dx\,dy+\iint_{D_2} x^2\,dx\,dy=\int_0^1\left(\int_{x^2}^1 y\,dy\right)dx+\int_0^1\left(\int_0^{x^2} x^2\,dy\right)dx=$$

$$=\int_0^1\left[\frac{y^2}{2}\right]_{y=x^2}^{1}dx+\int_0^1\left[x^2y\right]_{y=0}^{x^2}dx=\int_0^1\left(\frac{1}{2}-\frac{x^4}{2}+x^2\cdot x^2-x^2\cdot 0\right)dx=\int_0^1\left(\frac{x^4}{2}+\frac{1}{2}\right)dx=\left[\frac{x^5}{10}+\frac{x}{2}\right]_0^1=$$

$$=\frac{1}{10}+\frac{1}{2}=\frac{3}{5}$$

6.8a) $I=\int_0^1\Bigg(\underbrace{\int_{x^{1/3}}^1\frac{dy}{\sqrt{1+y^8}}}_{\text{"omöjlig" primitiv}}\Bigg)dx=\iint_D\frac{dx\,dy}{\sqrt{1+y^8}}$ $\quad D:\begin{cases} 0\le x\le 1 \\ x^{1/3}\le y\le 1 \end{cases}$



Byt integrationsordning, D kan skrivas $\begin{cases} 0\le y\le 1 \\ 0\le x\le y^3 \end{cases}$

$$\Rightarrow I=\int_0^1\left(\int_0^{y^3}dx\right)\frac{dy}{\sqrt{1+y^8}}=\int_0^1\left[x\right]_{x=0}^{y^3}\frac{dy}{\sqrt{1+y^8}}=\int_0^1\frac{y^3}{\sqrt{1+y^8}}dy=\Big/\begin{matrix} t=y^4 & y=0\Rightarrow t=0 \\ dt=4y^3dy & y=1\Rightarrow t=1 \end{matrix}\Big/=$$

$$=\frac{1}{4}\int_0^1\frac{dt}{\sqrt{1+t^2}}=\frac{1}{4}\left[\ln|t+\sqrt{1+t^2}|\right]_0^1=\frac{1}{4}\left(\ln(1+\sqrt{2})-\underbrace{\ln 1}_{=0}\right)=\frac{1}{4}\ln(1+\sqrt{2})$$